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PHYSICAL ARITHMETIC.

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"Every quantity is expressed by a phrase consisting of two components, one of these being the name of a number, and the other the name of a thing of the same kind as the quantity to be expressed, but of a certain magnitude agreed on among men as a standard or unit."

—CLERK-MAXWELL, *Heat*, p. 75.

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I. M
Irving Stringham

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PREFACE.

THIS book may be described as a treatise on applied arithmetic, the applications being chiefly in physical science. Knowledge of the elements of pure arithmetic is assumed, but the more advanced methods are explained when their application happens to occur.

The progress of physical science has caused the idea of the *unit* to become more prominent in text-books on arithmetic; and the old form of rule of three has been replaced to a large extent by what is called the unitary method, or the method of reduction to the unit. That method, in my opinion, very imperfectly represents the reasoning process involved.

The method developed in this work may be called the *equivalence* method. Each quantity is analysed into unit, numerical value, and, when necessary, descriptive phrase. The rate, or law, or convention, according to which one quantity depends on one or more other quantities, is expressed by an equivalence. These equivalences are of two kinds, absolute and relative—the former expressing the equivalence of *dependence*, the latter the equivalence of *substitution* or *replacement*. Finally equivalences are combined according to a form which is a development of the Chain Rule.

The present work is a development of notes which I began to take when a student on the subject of units and the reasoning processes involved in elementary calculations. My experience as examiner has confirmed me in the opinion that an elementary

work on the subject is a desideratum. In the carrying out of this rather laborious task, I have received much encouragement from Professor Tait and from Professor Chrystal, who has pointed out the want of a text-book on physical arithmetic in his reports as examiner of High Schools.

The writings which I have studied most in this connection are those of Clerk-Maxwell, Sir W. Thomson, Professor Tait, and Professor Everett; while for numerical data I have consulted principally the Tables of Landolt and Börnstein. I have gone on the principle of comparing the values given by the different authorities so as to retain the common figures, and cut off the uncertain remainder. As a general rule every one of the figures given may be relied on as significant, with the exception of perhaps the last.

The examples have been partly selected from the recent examination papers of the several Universities, and partly prepared by myself either for examinations or to illustrate kinds of calculation which were otherwise not illustrated.

ALEXANDER MACFARLANE.

EDINBURGH, 31st July, 1884.

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PHYSICAL ARITHMETIC.

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CHAPTER FIRST.

FINANCIAL.

SECTION I.—VALUE.

ART. 1.—**Standard of Value.** Every currency is based upon a standard of value, that is, a definite quantity of some precious substance. Gold and silver are very suitable substances for the purpose, consequently they have been adopted in almost all parts of the world. Gold may be adopted as the sole standard, or silver as the sole standard, or both may be adopted, forming what is called a double standard. A double standard is made possible by fixing by law the value of silver in terms of gold; but as the commercial value of silver relatively to gold is subject to fluctuation, it may be necessary to change the legal value from time to time, to prevent the metal which is undervalued from being exported to other countries. Formerly the standard of this country was double, but since 1816 gold has been the sole standard. Silver is at present the standard in India. Until recently the standard was double in France, the relative value of gold to silver being fixed at $15\frac{1}{2}$ to 1; but the adoption by Germany in 1871 of a single gold standard has caused France to adopt a single gold standard also.

ART. 2.—**Principal Unit of Value.** The principal unit of value, in the monetary system of this country, is the amount of pure gold contained in the sovereign. The sovereign is 123·27447 grains in weight, and eleven twelfths of this is pure gold; the remaining twelfth is alloy, which is disregarded in estimating the value of the coin. This unit of value is denominated the *pound sterling*.

The pound sterling was originally a pound weight of silver; the pound used being neither the troy nor the avoirdupois, but the pound sterling, which was equivalent to 5400 grains. A standard of this pound was preserved at one time in the Tower of London, hence the denomination of a "Tower pound." The value of the pound was depreciated from time to time, so that the term does not now at all agree with its ancient definition.

ART. 3.—**Units of Account.** The units in terms of which accounts are kept are the pound, the shilling, and the penny. The shilling, considered not as a coin, but as a unit of value, is the twentieth part of the pound; the penny, considered as a unit of value, is the twelfth part of the shilling so defined, or the two hundred and fortieth part of the pound. The farthing as used in accounts is rather a fraction of the penny than a separate denomination; it is not taken account of by public offices, bankers, and merchants. The abbreviations L. S. D. Q. are the initial letters of the Latin words *libra*, *solidus*, *denarius*, *quadrans*.

ART. 4.—**Metallic Currency—Gold Coins.** A coin proper is a stamped piece of metal, the stamp guaranteeing that the piece contains a specified amount of the standard metal. Hence in the system of this country the coins proper are of gold—they are the five-pound piece, the two-pound piece, the sovereign, and the half-sovereign, but the two former are not in general circulation. The fineness is the same for all, namely, eleven twelfths pure gold to one twelfth of alloy. This fineness may be expressed millesimally

as 916·66 parts by weight of pure gold in 1000 parts by weight of the coin. The weights of the coins are in proportion to their values.

The substance of a coin is slowly diminished by ordinary wear and tear, and it might be greatly diminished for fraudulent purpose; hence in the case of coins proper it is necessary to fix the minimum weight at which the coin will be accepted at its full value. This is called the *least current weight.* For a sovereign it is 122·50000 grains, which differs from the standard weight by ·77447 grains. The least current weight for the half-sovereign is 61·125 grains. The lightness of coin can only be determined by weights which have been examined and certified by the Board of Trade. When gold coins are tendered at the Bank of England each piece is weighed singly, and the light coins are cut and returned to the person tendering them, who has to bear whatever they have lost by abrasion during their existence as coins.

ART. 5.—**Silver Coins.** The silver coins at present in use are, *first*, the florin, equivalent to two shillings, the shilling, the sixpence, and the threepence; *second*, the crown, equivalent to five shillings, and the half-crown, neither of which have been struck since 1851, and the groat, or fourpence, which has not been struck since 1856; *third*, the twopence and the penny, which are only coined for the purpose of being distributed (with the groat and threepence) as alms by the sovereign on the Thursday before Easter, and are on that account called "maundy money." The fineness of all these silver coins is the same, and is thirty-seven fortieths by weight of pure silver to three fortieths by weight of alloy. The standard weight of the shilling is 87·27272 grains, or 66 shillings are coined out of 1 lb. troy (5,760 grains) of silver of the standard fineness. The weights of the other coins are in proportion to their nominal values.

ART. 6.—**Legal Tender; Token-coin.** Silver coin are *legal tender*

for the payment of an amount not exceeding forty shillings, but for no greater amount. The intrinsic value of a shilling coin varies with the price of silver in terms of gold, but it is always considerably less than the twentieth part of a pound. It is about the one twenty-fifth part. Hence the need for the limitation mentioned. Silver is more suitable than gold for coins of less value than the half-sovereign, because gold coins truly representative of the values would be inconveniently small. As the silver coins are not full equivalents of what they pass for, they are called *token*-coins.

ART. 7.—**Bronze Coins.** The bronze coins (introduced in 1860) comprise the penny, halfpenny, and farthing. They are made of a bronze alloy, the composition of which is 95 parts by weight of copper to 4 of tin and 1 of zinc. A pound avoirdupois of the bronze is coined into 48 penny pieces, or 80 halfpenny pieces, or 160 farthings. Thus 3 pennies, or five halfpennies, or 10 farthings weigh one ounce avoirdupois. These coins are also made so as to serve as measures of length. The diameter of the penny is one tenth of a foot, that of the halfpenny is an inch, and that of the farthing four fifths of an inch.

Bronze coins are legal tender for the payment of an amount not exceeding one shilling, but for no greater amount. Their intrinsic value is very much less than their nominal value, so that they are tokens rather than coins. The intrinsic value of the penny is about one fourth of a penny.

There is no least current weight for either silver or bronze coins. The light ones are selected when passing through the Bank of England, and the loss involved in re-coining is borne by the Government.

ART. 6.—**Paper Currency.** A bank note is a promise on the part of the bank to pay on demand the sum specified on the note. Notes for £5, £10, £20, £50, £100, £500, and £1000 are issued by the Banks of England, Scotland, and Ireland, and notes for £1

by the Banks of Scotland and Ireland. Bank of England notes are a legal tender for any sum, except at the Bank and in Scotland and Ireland. The issue of notes by a bank is restricted by law to a fixed amount depending upon the securities held by the bank; any further issue of notes must be represented by an equivalent amount of *specie* (that is, coined or uncoined gold or silver) in the head-office of the bank. Of this specie one fifth may be silver.

ART. 9. —**Equivalence.** The definitions of the units *farthing*, *penny*, and *shilling* are usually written in the form

$$\begin{aligned} 4 \text{ farthings} &= 1 \text{ penny}, \\ 12 \text{ pence} &= 1 \text{ shilling}, \\ 20 \text{ shillings} &= 1 \text{ pound}. \end{aligned}$$

These are not equations, but equivalences. In an equation the unit is the same for both sides, and it is not necessary that it should be expressed explicitly; in an equivalence the units are different and require to be explicitly expressed.

An equivalence remains true when both sides are multiplied by the same number, or both are divided by the same number. From this follows a rule for elimination, when one unit is given in terms of a second, and that second in terms of a third. Take, for example,

$$\begin{aligned} 12 \text{ pence} &= 1 \text{ shilling}, \\ 20 \text{ shillings} &= 1 \text{ pound}. \end{aligned}$$

From these two equivalences we deduce the equivalence

$$12 \times 20 \text{ pence} = 1 \times 1 \text{ pound},$$

i.e., $$240 \text{ pence} = 1 \text{ pound}.$$

In the same way, from

$$\begin{aligned} 4 \text{ farthings} &= 1 \text{ penny}, \\ 6 \text{ pence} &= \tfrac{1}{2} \text{ shilling}, \\ 2 \text{ shillings} &= \tfrac{1}{10} \text{ pound}, \end{aligned}$$

we deduce $$4 \times 6 \times 2 \text{ farthings} = 1 \times \tfrac{1}{2} \times \tfrac{1}{10} \text{ pound},$$

i.e., $$960 \text{ farthings} = 1 \text{ pound}.$$

ART. 10.—**Chain Rule.** The rule may be stated as follows:— Arrange the equivalences so that the consequent unit of the uppermost shall be the antecedent unit of the second, and the consequent unit of the second the antecedent unit of the third, and so on for any number of equivalences; then the unit on the left-hand which does not appear on the right-hand side multiplied by all the multipliers on its own side is equivalent to the unit on the right-hand side which does not appear on the left-hand side, multiplied by all the multipliers on its own side. This rule is called the "Chain Rule," and its application is very extensive in physical arithmetic.

ART. 11.—**Reciprocal.** Each equivalence, such as

$$20 \text{ shillings} = 1 \text{ pound},$$

has a reciprocal form. That of the equivalence mentioned is

$$\tfrac{1}{20} \text{ pound} = 1 \text{ shilling},$$

or

$$\cdot 05 \text{ pound} = 1 \text{ shilling}.$$

The number indicated by $\frac{1}{20}$, namely ·05, is called the reciprocal of the number 20. The value of the reciprocal rate is the reciprocal of the value of the primary rate.

It has become customary among scientific writers, after the example of Stokes, to write the fraction $\frac{1}{20}$ in the form 1/20, especially when the fraction occurs not in an equation but in the text. The slant bar is more convenient than the horizontal bar in several respects.

ART. 12.—**Conversion.** The simplest kind of reduction is called conversion; it consists in changing from a single unit to another single unit. Example: convert £123 into shillings.

$$20 \text{ shillings} = 1£,$$

$$123£,$$

$$\therefore \quad 123 \times 20 \text{ shillings},$$

$$\textit{i.e.,} \quad 2460 \quad ,,$$

Again, to convert 456 shillings into pounds :

$$
\begin{array}{rl}
 & 20 \text{ shillings} = 1\pounds, \\
 & \qquad\qquad 456 \text{ shillings}, \\
\therefore & \tfrac{456}{20}\pounds, \\
\textit{i.e.}, & 22{\cdot}8\pounds.
\end{array}
$$

The expression is wanted entirely in terms of the pound, not as £22 16*s.*

The Chain Rule applies to calculations of this kind; when there is only one odd unit we get a quantity and not an equivalence as the result of the reasoning. In a simple application of the Chain Rule it is not necessary to repeat the unit the second time; it may be understood. But it is important to arrange each unit in strict alternate order.

ART. 13.—**Reduction.** In reduction we are given a quantity expressed in a plurality of units, and are required to express it in terms of one unit; or we are given the quantity expressed in terms of one unit, and are required to express it in terms of a plurality. Example: Reduce £97 16*s.* $5\frac{3}{4}d.$ to farthings.

$$
\begin{array}{rr}
 & \pounds 97 \;\; 16 \;\; 5\tfrac{3}{4} \\
 & 20 \\
\hline
 & 1956 \\
 & 12 \\
\hline
 & 23477 \\
 & 4 \\
\hline
\textit{Answer}, & 93911 \text{ farthings.}
\end{array}
$$

Here we have multiplied the pounds by $20 \times 12 \times 4$, the shillings by 12×4, and the pence by 4. These three operations are combined, so far as they permit, to expedite the calculation.

Again, to express the same quantity in terms of the pound,

$$
\begin{array}{r|l}
4 & 3 \\
\cline{2-2}
12 & 5{\cdot}75 \\
\cline{2-2}
20 & 16{\cdot}48 \\
\cline{2-2}
 & 97{\cdot}824
\end{array}
$$

Here we have divided the 3 by $4 \times 12 \times 20$, the 5 by 12×20, the 16 by 20; and the operations have been combined so far as possible for the purpose of accelerating the computation.

An example of the second kind of question is: Express 84783 farthings in terms of pounds, shillings, pence, and farthings.

$$\begin{array}{r|l} 4 & 84783 \\ \hline 12 & 21195\frac{3}{4} \\ \hline 20 & \;1766 \quad 3\frac{3}{4} \\ \hline & \;\text{£}88 \quad 6 \quad 3\frac{3}{4} \end{array}$$

This is equivalent to dividing 84783 by $20 \times 12 \times 4$ to get the number of pounds, dividing the remainder by 12×4 to get the number of odd shillings, and dividing the remainder of the remainder by 4 to get the number of odd pence.

The laborious processes of common reduction arise entirely from the fact that a quantity is expressed, not in terms of one unit and its decimal derivatives, but in terms of a series of units related by numbers which have no system in themselves, and are not identical with or multiples or sub-multiples of 10, the base of the notation of Arithmetic.

ART. 14.—**Decimalization of Money.** It is very important to have a ready rule for transforming any sum expressed in terms of *l. s. d.* into one expressed in terms of *l.* alone. This is done by taking *first* the shillings, and *second* the pence and farthings reduced to farthings.

First. Since $2s. = \frac{1}{10}l.$, the number of the shillings when even divided by 2 will give the number of tenths; when odd, the number of tenths and in addition five hundredths.

Second. 1 farthing $= \frac{1}{960}$ pound

$$= \left(\tfrac{1}{1000} + \tfrac{1}{960} - \tfrac{1}{1000}\right) \text{ pound}$$

$$= \left(1 + \tfrac{1}{24}\right) \tfrac{1}{1000} \text{ pound.}$$

Hence the number of pence and farthings reduced to farthings

will be very nearly the number of thousandths of a pound. When the number of farthings is greater than 24 (that is, between 24 and 48), the correction amounts to 1, and 1 is to be added to the number of thousandths.

Example 1. Express $14s.\ 7\frac{1}{2}d.$ in terms of $l.$

$$14s. = \cdot 7l.$$
$$30f. = \cdot 031l.$$
$$\therefore\quad 14s.\ 7\tfrac{1}{2}d. = \cdot 731l.$$

Example 2. Decimalize $15s.\ 2\frac{3}{4}d.$

$$15s. = \cdot 75l.$$
$$11f. = \cdot 011l.$$
$$\therefore\quad 15s.\ 2\tfrac{3}{4}d. = \cdot 761l.$$

The converse rule is easily deduced. Take the number of tenths and ·5 from the hundredths (if the hundredths amount to five), that number multiplied by 2 gives the number of shillings; take the remaining numbers as referring to one thousand, they give the number of pence and farthings, but are to be diminished by 1 if the number lies between 25 and 50.

ART. 15.—**Proposed Decimal System of Units of Value.** The Committee on Decimal Coinage in the report which they presented to the House of Commons in 1853 recommended a system of units of value based upon the above method of decimalizing English money. They proposed to retain the pound as the primary unit, to retain florin as the name of the tenth part, to introduce a new term "cent" for the hundredth part, and a new term "mil" for the thousandth part. So that we should have the decimal equivalences

$$1l. = 10 \text{ florins} = 100 \text{ cents} = 1000 \text{ mils}.$$

All coins were to be equal to, or simple multiples of, these units.

The reform failed to be carried at the time. Since then a wider question has been raised—that of a uniform international system of units of value, or the modification of the primary units of value of the chief countries supposed founded on gold of

the same fineness, so that they might have simple relations to one another. The equivalences 1 pound = 25 francs, 1 dollar = 5 francs, 1 Austrian florin = 2·5 francs are approximately true; and if they were made exactly true, calculations of exchange would be greatly facilitated.

With a decimal system of units, reduction can be done without computation; thus

$$12{\cdot}345l. = 12l.\ 3fl.\ 4c.\ 5m. = 12345m.$$
$$= 123fl.\ 45m. = 12l.\ 34{\cdot}5c.$$

Here *fl.*, *c.*, and *m.* are used as abbreviations for florin, cent, and mil.

EXERCISE I.

1. Express 10,000 halfpence in terms of the pound, and 978*l.* in terms of the farthing.

2. Express 9*l.* 17*s.* 8½*d.* in terms of the shilling alone, and express 1234·56 shillings in terms of *l. s. d.*

3. Decimalize 2*l.* 15*s.* 7*d.* and 8*l.* 1*s.* 5½*d.*

4. Express in *l. s. d.* 12·186*l.* and 17·914*l.*

5. Express in terms of the proposed decimal units 11*l.* 7*s.* 6½*d.* and 7*l.* 10*s.* 9*d.* and 8*l.* 15*s.* 6*d.*

6. Express in terms of the present units 7*l.* 5*fl.* 3*c.* 7*m.* and 45*l.* 3*fl.* 9*c.* 8*m.*

7. Express in *l. s. d.* 014*l.*, 141*l.*, 1·419*l.*, 14·193*l.*

8. Find the equivalents in the proposed decimal system of (1) crown, (2) half-crown, (3) sixpence, (4) fourpence, (5) threepence.

9. Find the equivalents of 4 mils, 10 mils, 20 mils, 125 mils, 200 mils.

10. Express 45·382*l.* in terms of *l. s. d.*

11. Express 98*l.* 17*s.* 6¾*d.* as the decimal of a pound, and 87*l.* 16*s.* 5¼*d.* as the decimal of a shilling.

SECTION II.—PRICE.

ART. 16.—**Discrete, Continuous.** A commodity may be either discrete or continuous. It is said to be discrete when it is composed of a number of organic or manufactured units, as eggs; it is said to be continuous when it is not made up visibly of units, as

butter. A discrete commodity can be counted; a continuous commodity must either be weighed or measured. When a discrete commodity is composed of a great number of units, as wheat, it also must be weighed or measured. Weighing is a more exact method of meting out than measuring.

ART. 17.—**Rate of Price.** By the *price* of a commodity is meant its value estimated in terms of a recognised unit of value. By the *rate* of price of a commodity (usually called the price) is meant the equivalent in the unit of value of a unit of the commodity. The values of the rates for the different kinds of commodities which hold for a particular district and interval of time are published in the market tables. For example—

Eggs,	- -	14 pence per dozen.
Butter,	- -	16 pence per lb.
Bread,	- -	7 pence per loaf of 4 lbs.
Wheat,	- -	6 shillings per bushel.

Frequently the unit of the commodity is not mentioned, as the customary unit is well understood.

A rate is an equivalence. The above prices can be expressed as—

$$\begin{aligned}
14 \text{ pence} &= 1 \text{ dozen eggs.} \\
16 \text{ pence} &= 1 \text{ lb. butter.} \\
7 \text{ pence} &= 1 \text{ loaf of 4 lbs.} \\
6 \text{ shillings} &= 1 \text{ bushel wheat.}
\end{aligned}$$

Here = is read "for every."

ART. 18.—**Transformation of Price.** The equivalence expressing a price may be capable of application only within certain limits; that is, it may be true for all quantities of the commodity not exceeding a certain amount. The equivalence may have a different value for quantities of the commodity exceeding that amount. But this is not to be considered as invalidating the statement that an equivalence is unaltered when both sides are

multiplied by the same quantity; it is a matter which affects not the value of the rate, but its application to a particular case.

If it is required to transform $15d.$ per lb. into $l.$ per ton, the process may be arranged as follows—

$$1l. = 20 \times 12d.$$
$$15d. = 1 \text{ lb.}$$
$$28 \times 4 \text{ lb.} = 1 \text{ cwt.}$$
$$20 \text{ cwt.} = 1 \text{ ton};$$
$$\therefore \quad 15 \times 28 \times 4 \times 20l. = 20 \times 12 \text{ ton},$$
$$\textit{i.e.,} \quad 280l. = 1 \text{ ton}.$$

This arrangement facilitates both reasoning and cancelling.

ART. 19.—**Barter.** From the price-rates of two commodities we can deduce the barter-rate between them. We have merely to eliminate the common unit of value by an application of the Chain Rule. For example—

Given 14 pence = 1 dozen eggs,
and 1 lb. of butter = 16 pence;
it follows that 14 lb. of butter = 16 dozen eggs,
or 7 lb. of butter = 8 dozen eggs,
or $\frac{7}{8}$ lb. of butter = dozen eggs.

ART. 20.—**Price depending on Time.** A commodity which is consumed by each of a number of agents in equal quantities during equal intervals of time, may have its value expressed in terms of an agent and a unit of time. Thus, for lamps of equal power, the price of gas may be stated at $2d.$ per lamp per hour. Here we have two independent quantities—that is, quantities upon which the first-mentioned quantity depends.

Similarly, when work is performed uniformly by a number of workmen, the price of the labour and their rate of pay may be expressed in terms of a workman and a unit of time. For example, the wages of a class of workmen may be $6\frac{1}{2}d.$ per man per hour.

A rate of this form can also be expressed as an equivalence; thus—

$$2d. = 1 \text{ lamp by hour},$$
$$2d. \text{ per lamp} = 1 \text{ hour},$$
$$2d. \text{ per hour} = 1 \text{ lamp}.$$

ART. 21.—**Price depending on Space.** In the case of carrying goods, the service rendered is sometimes taken as proportional to the mass (weight) of the goods transferred, and also to the distance over which they have been carried. Thus a railway rate for goods carried may be $\frac{1}{4}d.$ per cwt. per mile. The charge for passengers is also proportional to the distance, but only two weights are recognised—an adult and a juvenile. For example, the parliamentary fare is 1*d.* per adult passenger per mile.

When the rate for carrying introduces the distance, there may be considerable trouble in its application. There is first to ascertain the distance in a given case, and then there is the extra calculation involved after the number has been found. Hence the great practical convenience of making such a rate as the rate of postage independent of the distance the letter has to be carried, provided that the distance is within a well-defined region, such as the United Kingdom. As regards the other variables, a letter is not charged according to its weight, but in accordance with a fixed scale of rates.

For book-post we have the rate

$$1 \text{ halfpenny} = 2 \text{ oz.};$$

but in application a fraction of 2 oz. is to be considered equivalent to an additional 2 oz., and the weight must not be greater than 5 lbs. The maximum weight is thus 40×2 oz., and therefore the maximum charge 1*s.* 8*d.* Here we have an example of a rate which is *discontinuous*, that is, proceeds by leaps, and the whole range also is limited.

ART. 22.—**Change of Price.** Change of price may be expressed

in the same form as the price, or it may be expressed as a percentage on the former price.

Suppose that the old price is

$$n \text{ shillings} = \text{lb.} \qquad (1)$$

the change may be

$$a \text{ shillings advance} = \text{lb.}$$

or

$$a \text{ shillings reduction} = \text{lb.};$$

therefore the new price is

$$n \pm a \text{ shillings per lb.} \qquad (2)$$

From (1) and (2) we deduce

$$n \pm a \text{ shillings new price} = n \text{ shillings old price.}$$

Suppose that the change given in the form of a percentage is 10 per cent. advance.

This means

$$10 \text{ shillings advance} = 100 \text{ shillings former price,}$$

or

$$\tfrac{1}{10} \text{ shilling advance} = \text{shilling former price.}$$

Hence

$$1 + \tfrac{1}{10} \text{ shilling new price} = \text{shilling old price;}$$

and the new price is

$$n(1 + \tfrac{1}{10}) \text{ shilling per lb.}$$

EXAMPLES.

Ex. 1.—Find the value of the wheat crop grown on 89 acres, when the average yield is 31 bushels per acre, and the price of wheat is 6*s.* 3*d.* per bushel.

$$25 \text{ shillings} = 4 \text{ bushels,}$$
$$31 \text{ bushels} = 1 \text{ acre,}$$
$$89 \text{ acres;}$$

$$\therefore \quad \frac{25 \times 31 \times 89}{4} \text{ shillings, } i.e., \text{ £}862l.\ 3s.\ 9d.$$

Ex. 2.—What do eggs cost per dozen when 12 fewer in a shilling's worth raises the price 2*d.* per dozen ?

Let real price be x pence $=$ 1 dozen eggs,

$$\therefore \quad 12 \text{ pence} = \frac{12}{x} \text{ dozen eggs};$$

$$\therefore \quad \text{supposed price is 12 pence} = \left(\frac{12}{x} - 1\right) \text{ dozen eggs},$$

or
$$\frac{12}{\frac{12}{x}-1} \text{ pence} = 1 \text{ dozen eggs}.$$

But supposed price is also $x + 2$ pence $=$ 1 dozen eggs.

Hence
$$\frac{12}{\frac{12}{x}-1} \text{ pence per dozen} = x + 2 \text{ pence per dozen}.$$

Here we have an equation in which the unit is the same on the two sides; it can therefore be solved on general principles, independently of the unit. It reduces to the quadratic equation

$$x^2 + 2x - 24 = 0;$$

hence $x = 4$ or -6. It is the positive value which is the answer to our question; hence the answer is 4 pence per dozen.

Ex. 3.—*A* and *B* together spend a shilling on 65 apples, *A*'s costing six a penny, and *B*'s 20 per cent. more. How many did each buy?

Price paid by *A*, 1 penny $=$ 6 apples.
Additional price by *B*, 20 pence additional $=$ 100 pence by *A*,
i.e., $\frac{1}{5}$ penny additional $=$ penny by *A*,
$\therefore$ price by *B*, $(1 + \frac{1}{5})$ penny $=$ 6 apples,
i.e., 1 penny $=$ 5 apples.

Suppose *A* bought x apples; then he spent $\frac{x}{6}$ pence. And *B* bought $65 - x$ apples; $\therefore$ he spent $\frac{65-x}{5}$ pence.

Hence
$$\frac{x}{6} + \frac{65-x}{5} = 12;$$

from which $x = 30$. Hence *A* bought 30 apples, and *B* 35.

Ex. 4.—If 3 ducks be worth 4 chickens, and 2 geese be worth 7 ducks, find the value of a goose when a pair of chickens can be bought for 3*s.* 9*d.*

$$\begin{aligned} 2 \text{ chickens} &= 3\tfrac{3}{4} \text{ shillings}, \\ 3 \text{ ducks} &= 4 \text{ chickens}, \\ 2 \text{ geese} &= 7 \text{ ducks}, \\ & 1 \text{ goose}. \end{aligned}$$

$$\therefore \quad \frac{4 \times 7 \times 3\frac{3}{4}}{2 \times 3 \times 2} \text{ shillings, } i.e., \ 8s.\ 9d.$$

Ex. 5.—There are two coins such that 15 of the first and 14 of the second have the same value as 45 of the first and 6 of the second. What is the ratio of the value of the first coin to that of the second?

Let V denote the first coin, and V′ the second. Then we have

$$\begin{aligned} 15\,V + 14\,V' &= 45\,V + 6\,V', \\ \therefore \quad 8\,V' &= 30\,V, \\ \text{or} \quad 4\,V' &= 15\,V. \end{aligned}$$

Thus 4 of the second coin are equivalent (in value) to 15 of the first.

Ex. 6.—Find the cost of electric lighting for 13 weeks, with sixty lamps burning on an average eight hours per day, the rate of charge being $2\frac{1}{4}d.$ per lamp per hour.

$$\begin{aligned} 2{\cdot}25 \text{ pence per hour} &= 1 \text{ lamp}. \\ & 60 \text{ lamps}; \\ \therefore \quad 60 \times 2{\cdot}25 \text{ pence} &= 1 \text{ hour}, \\ 8 \text{ hours} &= 1 \text{ day}, \\ 7 \text{ days} &= 1 \text{ week}. \\ & 13 \text{ weeks}; \end{aligned}$$

$$\begin{aligned} &\therefore \quad 60 \times 2{\cdot}25 \times 8 \times 7 \times 13 \text{ pence}, \\ &i.e., \quad 98{,}280 \text{ pence}, \\ &i.e., \quad 409l.\ 10s. \end{aligned}$$

EXERCISE II.

1. Find the value of a steel hammer weighing 225 tons at the rate of $1\frac{3}{4}$ lbs. for $3d.$

2. Find the value of 1000 lbs. pimento sold at $3\frac{1}{3}\frac{3}{2}d.$ per lb.

3. Three thousand workmen strike because an advance of one halfpenny per hour is not conceded. Find the additional sum which would be required weekly to concede the demand, taking a working day to consist of nine hours.

4. A burner consuming 4 cubic feet of gas per hour is used on an average 6 hours per day during a year. Required the cost at $4s.$ $6d.$ per 1000 cubic feet.

5. If, for this country, we have on an average eighteen letters per head per annum, find the average number of letters for a month; and also the revenue for the month, taking the average postage at one penny halfpenny per letter. The population is 35,262,762.

6. Find the value of 215 tons 17 cwt. 3 qrs. 9 lbs. of cast-iron at $9l.$ $11s.$ $6\frac{1}{4}d.$ per ton.

7. Calculate to the nearest farthing, by whatever process you consider the most convenient, the cost of 19 tons 19 cwt. 3 qrs. $27\frac{1}{2}$ lbs. of merchandise at $19l.$ $19s.$ $11\frac{1}{2}d.$ per ton.

8. A farmer sold seven oxen and twelve cows for $250l.$ He sold three more oxen for $50l.$ than he did cows for $30l.$ Required the price of each.

9. If 21 men in 27 days receive $141l.$ $15s.$ in wages, how many men must work 21 days to earn $157l.$ $10s.$?

10. If it cost $61l.$ $18s.$ $5d.$ to keep 2 horses for 11 months, how long can 3 horses be kept for $59l.$ $2s.$ $1\frac{1}{2}d.$?

11. How far ought $4\frac{3}{4}$ cwt. to be carried for $11s.$ $10\frac{1}{2}d.$ when the carriage of $17\frac{1}{2}$ cwt. for 52 miles cost $8s.$ $4d.$?

12. If 64 cwt. carried 40 miles cost $10s.$ $8d.$, what will 80 cwt. carried 63 miles cost?

13. A reduction of 20 per cent. in the price of beef would enable a purchaser to obtain 6 lbs. more for a sovereign. What is the reduced price?

14. What are ashes per 100 loads when 8 more loads for a sovereign lower sthe price a penny a load?

15. What are eggs selling at when, if they be raised threepence the dozen, one would get four fewer in a shilling's worth?

16. A reduction of 30 per cent. in the price of eggs would enable a purchaser to obtain 54 more for a guinea. What may the present price be?

17. A reduction of 10 per cent. in the price of iron would enable a purchaser to obtain one hundredweight more for a sovereign. What is the present price?

18. "There is not one of those countries where they don't pay 30 to 40 per cent. more for sugar than you would pay in England." Find the approximate foreign price when the home price is 5d. per lb.

19. In a mixed excursion train of 1000 passengers, the fares in the three classes are in the ordinary proportion of 4, 3, 2; and the corresponding receipts are $52l.$,

57*l*., and 61*l*. respectively. Required the fare and the number of passengers in each class.

20. *A* gets half of *B*'s money, and then *B* gets half of *A*'s ; *B* then has 20 per cent. more than *A*. What relative sums had they at first?

21. How many lbs. of butter at 1*s*. $3\frac{1}{2}d$. per lb. must a farmer give to a grocer for 8 lbs. of sugar at $5\frac{3}{4}d$. per lb.?

22. A farmer sold on one day 2 oxen and 3 sheep for 34*l*. 10*s*., 4 oxen and 5 sheep for 67*l*. 10*s*., and 6 oxen and 7 sheep for 100*l*. Is it possible that all the oxen were sold at one price, and all the sheep at one price?

23. 14 tons 3 cwt. copper at 92*l*. 15*s*. per ton ; 12 tons 12 cwt. spelter at 35*s*. 10*d*. per cwt. ; 3 tons 5 cwt. tin at 78*s*. 9*d*. per cwt. Coal consumed, 14 tons at 14*s*. 3*d*. Loss of metal in casting $\frac{1}{30}$. Labour equal to one man for 31 days at 4*s*. 9*d*. Other expenses reckoned at 125*l*. What is the cost per ton to the nearest shilling?

SECTION III.—PROFIT AND LOSS.

ART. 23.—**Profit and Loss.** A merchant both buys and sells. Suppose that a grocer buys at the rate of

$$m \text{ pence cost} = \text{dozen eggs}, \qquad (1)$$

and sells at the rate

$$n \text{ pence receipt} = \text{dozen eggs}; \qquad (2)$$

from these two rates we deduce a third, viz. :—

$$n \text{ pence receipt} = m \text{ pence cost}. \qquad (3)$$

Again, from (1) and (2) we deduce the rate of profit or loss; it has three forms depending on the three quantities involved in the two rates—

$$(n-m) \text{ pence profit or loss} = \text{dozen eggs}, \qquad (4)$$

$$(n-m) \text{ pence profit or loss} = m \text{ pence cost}, \qquad (5)$$

$$(n-m) \text{ pence profit or loss} = n \text{ pence receipt}. \qquad (6)$$

If n is greater than m we have profit, if n is less than m we have loss.

Observe—The sign = in these equivalences means *for every*. When the number on the right-hand side of the equivalence is 1, it need not be expressed.

ART. 24.—**Percentage; Ratio.** Rate of profit or loss is usually given in the form of a percentage, and there is frequently an ambiguity arising from the fact that it is not mentioned whether it is on cost or receipt. In such case it generally refers to the cost. A profit of 12 per cent. on the cost means

12 pence profit per 100 pence cost;

or, 12 pence profit = 100 pence cost.

Any other unit of value may be substituted for the penny, because it occurs on both sides of the equation. A rate which has the same unit on the two sides is called a ratio-rate; the two units differ only in quality.

EXAMPLES.

Ex. 1. If by selling at 7*s.* 6*d.* per yard you lose 10 per cent. on the outlay, what do you gain or lose per cent. when you sell at 8*s.* 6*d.* per yard?

x pence cost = yard,

90 pence receipt = yard;

$\therefore$ $\frac{x-90}{x}$ pence loss = penny cost.

Now $\frac{10}{100}$ pence loss = penny cost,

$$\therefore \qquad \frac{x-90}{x} = \frac{1}{10}.$$

By solving this equation we get $x = 100$,

$\therefore$ 100 pence cost = yard.

Again 102 pence receipt = yard,

$\therefore$ $\frac{102-100}{100}$ pence profit = penny cost,

i.e., 2 pence profit = 100 pence cost,

i.e., 2 per cent. profit on outlay.

Ex. 2. A publisher sells books to a retail dealer at 5*s.* a copy, but allows 25 copies to count as 24. If the retailer sells each of the 25 copies for 6*s.* 9*d.*, what profit per cent. does he make?

$$24 \times 5 \text{ shillings cost} = 25 \text{ copies},$$

$$25 \text{ copies} = \frac{27 \times 25}{4} \text{ shillings receipt};$$

$$\therefore \frac{27 \times 25}{4} - 24 \times 5 \text{ shillings profit} = 24 \times 5 \text{ shillings cost},$$

i.e., $13 \quad ,, \quad = 32 \quad ,,$

$\therefore \quad \dfrac{13 \times 100}{32} \quad ,, \quad = 100 \quad ,,$

i.e., $41 -$ per cent.

Ex. 3. For a special sale a merchant gave his customers 40 per cent. off the marked price, but the goods had been marked at an advance of 60 per cent. on their cost. Did he gain or lose, and at what rate per cent. on the price received?

$$1 + \tfrac{6}{10}\text{£ marked} = \text{£ cost},$$

$$1 - \tfrac{4}{10}\text{£ receipt} = \text{£ marked};$$

$$\therefore \quad (1 + \tfrac{6}{10})(1 - \tfrac{4}{10})\text{£ receipt} = \text{£ cost},$$

i.e., $1 - \tfrac{4}{100}\text{£ receipt} = \text{£ cost};$

hence $\tfrac{4}{100}\text{£ loss} = 1 - \tfrac{4}{100}\text{£ receipt},$

$$1\text{£ loss} = 24\text{£ receipt},$$

$$4{\cdot}17\text{£} - \text{ loss} = 100\text{£ receipt}.$$

Thus he lost at 4·17 per cent.

Ex. 4. An article in passing from the producer to the consumer passed through the hands of three dealers, each of whom added for his own profit 10 per cent. on the price at which he bought. The final price was 365£; what was the original price?

$1 + \tfrac{1}{10}$£ charged by 1st dealer = £ paid to producer,

$1 + \tfrac{1}{10}$£ charged by 2nd dealer = £ paid to 1st dealer,

$1 + \tfrac{1}{10}$£ charged by 3rd dealer = £ paid to 2nd dealer,

365£ charged by 3rd dealer;

$$\therefore \quad \frac{365}{(1 + \frac{1}{10})^3}\text{£ paid to producer},$$

i.e., $\dfrac{365 \times 10^3}{11^3}$£ paid to producer

i.e., 274*l.* 4*s.* 7*d.* +.

Ex. 5. In shipping ice 12 per cent. is destroyed, 45 per cent. of the shipped ice melts during the passage, 20 per cent of the remainder is lost in landing. At what increase per cent. on the original price per lb. must the residue be sold in order to yield a profit of 142 per cent. on the whole ?

$$\begin{aligned} 88 \text{ lbs. shipped} &= 100 \text{ lbs. bought,} \\ 55 \text{ lbs. arrived} &= 100 \text{ lbs. shipped,} \\ 80 \text{ lbs. sold} &= 100 \text{ lbs. arrived;} \\ \therefore \quad 88 \times 55 \times 80 \text{ lbs. sold} &= 1{,}000{,}000 \text{ lbs. bought.} \end{aligned}$$

Let the buying price be x pence per lb., and the percentage of profit y pence $= 100$ pence paid, then the selling price is $x\left(1+\frac{y}{100}\right)$pence per lb. Hence

$$x\left(1+\frac{y}{100}\right)88 \times 55 \times 80 \text{ pence receipt} = x1{,}000{,}000 \text{ pence cost,}$$

$$i.e., \left(1+\frac{y}{100}\right)88 \times 55 \times 80 \text{ pence receipt} = 1{,}000{,}000 \text{ pence cost.}$$

$$\therefore \left(1+\frac{y}{100}\right)\frac{88 \times 55 \times 80}{1{,}000{,}000} - 1 \text{ pence profit} = \text{penny cost;}$$

but $$\frac{142}{100} \text{ pence profit} = \text{penny cost;}$$

$$\therefore \quad \left(1+\frac{y}{100}\right)\frac{88 \times 55 \times 80}{1{,}000{,}000} - 1 = \frac{142}{100}.$$

By solving this equation we get $y = 193$. The cost price must be increased by 193 per cent.

EXERCISE III.

1. A grazier bought a sheep for £1 6*s*., and sold it for £3, and incurred a loss equal to half the cost, plus a quarter of the expense of feeding. What was the expense of feeding?

2. Formerly newspaper wrappers were sold at the rate of eight for fourpence halfpenny, and now they are sold at twelve for sevenpence. What is the percentage increase on the former price?

3. A corn dealer bought wheat at £2 1*s*. 3*d*. per quarter, which he subsequently sold at £2 9*s*. 7*d*. per quarter, and made a profit of £277 10*s*. upon the transaction. How many quarters did he buy and sell?

4. A house costs the landlord £130 a year for rent and taxes. For one third of the year it is let at £6 10*s.* a week, and for the remainder of the year at £5 a week. Find the landlord's profit, and how much it is per cent.

5. A man purchases 80 oxen for £1000, and sells 24 of them at a loss of 3 per cent. outlay. At what rate per head must he sell the remainder so as to lose nothing on the whole?

6. An article is sold for 12*s.* at a loss of 4 per cent. on cost. At what price must it be sold that a gain of 4 per cent. may be made?

7. A person by selling at 4*s.* 1½*d.* per lb. an article which cost £21 per cwt., cleared 2 per cent. more profit than if he had sold the whole for £162. How much was sold of the article?

8. If by selling at 20*s.* you lose 16 per cent. on your outlay, at what rate do you sell when you gain 16 per cent. on your outlay?

9. If by selling at 15*s.* 6*d.* you lose 7 per cent. on the outlay, what do you gain or lose per cent. when you sell at 16*s.* 6*d.*?

10. A merchant buys cloth at 2*s.* 3¼*d.* per yard, and sells it at 3*s.* 4½*d.* per yard. What is the percentage of profit on the outlay; and how many yards must he sell to gain a profit of £10?

11. A person bought a lot of land for $10,000. He sold one half of it at a gain of 50 per cent., two fifths of it at $40 an acre, and the remainder at a loss of 40 per cent. He gained 45 per cent. on the whole. Find the number of acres in the lot.

12. A merchant receives £300 for sales in one day; on £200 he gains 40 per cent. What does he gain or lose per cent. on the remaining £100 in order that his profit for the day may be £50?

13. A dealer is paid £23 19*s.* 2*d.* for an article. Assuming that it passed through the hands of three dealers and that each added 10 per cent. of the price at which he bought for his own profit, what did the first dealer pay?

14. A man has 1000 apples for sale; at first he sells so as to gain at the rate of 50 per cent. on the cost price; when he has done this for a time the sale falls off, so he sells the remainder for what he can get, and finds that by doing so he loses at the rate of 10 per cent. If his total gain is at the rate of 29 per cent., how many apples did he sell for what he could get?

SECTION IV.—MIXTURE.

ART. 25.—**Composition.** Suppose that a tea-dealer forms a mixture out of three kinds of tea, which we shall distinguish as A, B, C. Suppose that he takes a lbs. of A, b lbs. of B, c lbs. of C and mixes them thoroughly. The composition of the resulting mixture is fully given by the equivalence

$$a \text{ lbs. of } A + b \text{ lbs. of } B + c \text{ lbs. of } C = a + b + c \text{ lbs. of mixture.} \quad (1)$$

From this complete equivalence certain partial equivalences may be derived; thus

$$a \text{ lbs. of } A = a + b + c \text{ lbs. of mixture,} \tag{2}$$

that is, there are a and only a lbs. of A, in every $a + b + c$ lbs. of mixture. (3)

Similarly, b lbs. of $B = a + b + c$ lbs. of mixture,

and c lbs. of $C = a + b + c$ lbs. of mixture. (4)

From these partial equivalences three of a different kind can be derived, namely,

$$a \text{ lbs. of } A = b \text{ lbs. of } B, \tag{5}$$

$$b \text{ lbs. of } B = c \text{ lbs. of } C, \tag{6}$$

and by immediate consequence from these two

$$c \text{ lbs. of } C = a \text{ lbs. of } A. \tag{7}$$

ART. 26.—**Price of Mixture.** Suppose that the cost price of A tea is p shillings per lb.; of B, q shillings per lb.; and of C, r shillings per lb. Then the cost of the a lbs. of A taken is pa shillings; of the b lbs. of B, qb shillings; of the c lbs. of C, rc shillings. Hence the resulting cost price of the mixture is

$$pa + qb + rc \text{ shillings} = a + b + c \text{ lbs. of mixture,}$$

or

$$\frac{pa + qb + rc}{a + b + c} \text{ shillings} = \text{lb. of mixture.}$$

If the cost price of the required mixture is known, say n shillings per lb., and the price of each of the components, but not the ratios of composition, we have for determining b/a and c/a the equation

$$\frac{pa + qb + rc}{a + b + c} = n,$$

i. e.,

$$\frac{p + q\frac{b}{a} + r\frac{c}{a}}{1 + \frac{b}{a} + \frac{c}{a}} = n.$$

If there are only two components, then $c/a = 0$, and the equation

gives the value of b/a. Otherwise the equation is indeterminate; it is then a question of finding the simplest solution.

For the meaning of the slant bar see Art. 11.

ART. 27.—**Sharing.** Denote a bankrupt's creditors by A, B, C, and their claims by a£, b£, c£, respectively. Then

$$a£ \text{ of } A + b£ \text{ of } B + c£ \text{ of } C = a + b + c£ \text{ liability.}$$

From this equivalence partial equivalences may be derived as in Art. 25.

It is a convention in the commercial world, that the assets are to be distributed according to the ratios supplied by the above equivalence.

EXAMPLES.

Ex. 1. A grocer mixes four kinds of tea, in the proportions of 1, 1·5, 2, 2·5 parts by weight of the several kinds respectively. Required the number of pounds of each kind in one hundredweight of the mixture.

2 lbs. 1st + 3 lbs. 2nd + 4 lbs. 3rd + 5 lbs. 4th = 14 lbs. mixture,

112 lbs. of mixture,

$\therefore$ $\dfrac{112 \times 2}{14}$ lbs. of 1st, *i.e.*, 16 lbs. of 1st.

Similarly 24 lbs. of 2nd, 32 lbs. of 3rd, 40 lbs. of fourth.

Ex. 2. The four kinds of tea in the preceding question, having cost the grocer at the rates of 5, 4, 3, 2 shillings per lb. respectively; required the price per lb. at which he must sell the mixture in order to realize 25 per cent. profit on his outlay?

2 lbs. 1st + 3 lbs. 2nd + 4 lbs. 3rd + 5 lbs. 4th = 14 lbs. mixture,

5*s*. per lb., 4*s*. per lb., 3*s*. per lb., 2*s*. per lb.

$\therefore$ 10 + 12 + 12 + 10 shillings cost = 14 lbs. mixture,

i.e., 22 „ = 7 „

Now $1 + \frac{1}{4}$ shilling receipt = 1 shilling cost;

$\therefore$ $(1 + \frac{1}{4})\ \frac{22}{7}$ shilling receipt = 1 lb. mixture;

i.e., $\frac{55}{14}$ „ = „

or 3*s*. $11\frac{1}{2}d$. per lb. mixture.

Ex. 3. Oranges are bought for half-a-crown a hundred; some are sold at 3*s.* 6*d.* a hundred, and the rest at 2*s.* 10½*d.* a hundred: the same profit is made as if they had all been sold at 3*s.* 1½*d.* a hundred. Of a thousand oranges sold, how many fetch 3*s.* 6*d.* a hundred?

x oranges 1st kind + $1000 - x$ oranges 2d kind = 1000 oranges mixed.

42 pence per 100 of 1st, 34·5 per 100 of 2nd,

$\therefore\ \frac{42x}{100} + \frac{34{\cdot}5\ (1000 - x)}{100}$ pence receipt = 1000 oranges mixed.

But this is equal to 37·5 pence receipt = 100 oranges mixed; therefore we get the equation

$$\frac{42x}{100} + \frac{34{\cdot}5\ (1000 - x)}{100} = 375,$$

which gives $\qquad x = 400.$

Hence 400 oranges out of 1000 sold fetch 3*s.* 6*d.* a hundred. Observe—It is not necessary to give the original price.

Ex. 4. Out of a cask containing 360 quarts of pure alcohol a quantity is drawn off and replaced by water. Of the mixture, a second quantity, 84 quarts more than the first, is drawn off and replaced by water. The cask now contains as much water as alcohol. Find how many quarts were taken out the first time. Show that the problem has only one solution.

x quarts water + $(360 - x)$ quarts alcohol = 360 quarts 1st mixture,

$(360 - x - 84)$ quarts 1st mixture left; therefore

$(360 - x - 84)\,\frac{x}{360}$ quarts water + $(360 - x - 84)\frac{360 - x}{360}$ quarts alc.

The second mixture is formed by filling up with water,

$\therefore\ (360 - x - 84)\,\frac{360 - x}{360}$ qrts. alcohol = 360 qrts. 2nd mixture.

But we are given $\qquad$ 1 quart alcohol = 2 qrts. 2nd mixture; hence the equation

$$(360 - x - 84)(360 - x) = 180 \times 360,$$

which reduces to the quadratic equation

$$x^2 - 636x + 96 \times 360 = 0,$$

the roots of which are $x = 576$ and $x = 60$. Only the latter is possible, for 576 is greater than 360.

Ex. 5. A creditor received 16*s.* 3*d.* in the pound, and thereby lost 135*l.* 10*s.*; how much was due to him?

$$16s. = \cdot 8£.$$

$$3d. = \cdot 0125£.$$

$$\therefore \quad 8125£ \text{ received} = 10000£ \text{ due,}$$

$$\therefore \quad 1875£ \text{ lost} \quad = 10000£ \text{ due,}$$

$$\therefore \quad \frac{135{\cdot}5 \times 10000}{1875} £ \text{ due,}$$

$$i.e., \quad \frac{271000}{375} £ \text{ due,}$$

$$i.e., \quad 722l.\ 13s.\ 4d.$$

EXERCISE IV.

1. A vessel is filled with a mixture of spirit and water in which there is 70 per cent. of spirit; 19 gallons are taken out, and the vessel is filled up again with water; the proportion of spirit is now found to be 56·7 per cent.; find how much the vessel contains.

2. If apples are bought for 4 a penny, and mixed with an equal number bought for 3 a penny, and then sold at the rate of 5 for 2 pence; what is the gain per cent. on the outlay?

3. If gunpowder is composed of nitre, charcoal, and sulphur, in the proportions of 16, 3, $2\frac{1}{4}$; how much of each is required for one cwt. of gunpowder?

4. An estate is divided into three portions of 250 acres 62 acres 2 roods, and 19 acres 1 rood 20 poles; these portions are let at 1*l.* 5*s.* 4*d.*, 1*l.* 1s. 8*d.*, and 3*l.* per acre respectively. At what uniform rent per acre might the whole estate be let so as to bring in the same rental?

5. A milk-dealer buys pure milk at $11\frac{1}{2}d.$ per gallon. How much water must he add that he may sell at 5*d.* a quart, and obtain a gross profit of 100 per cent.?

6. A tobacconist pays 4*s.*, 3*s.* 6*d.*, and 2*s.* 6*d.* per lb. for three kinds of tobacco. He mixes them, and, by selling at 4*s.* per lb., obtains a gross profit of 25 per cent. on his receipts. If he omitted the cheapest tobacco from his mixture, keeping the others in the same proportion, his profit would be only $2\frac{1}{2}d.$ per lb. What was his mixture?

7. The estate of a bankrupt pays 4*s.* $4\frac{1}{2}d.$ in the £; what loss will a creditor sustain whose claim is for 337*l.* 6*s.* 8*d.*?

8. A bankrupt owes 7,357*l.* 12*s.*, and his assets for distribution among his creditors amount to 3.065*l.* 13*s.* 4*d.* How much in the pound will they receive?

9. A bankrupt whose estate is worth 1,823*l* 18*s*. 9d. owes to three persons sums of 1,031*l*. 5s., 814*l*., and 586*l*. 13*s*. 4d. respectively; how much can he pay in the pound?

10. A statement of affairs showed liabilities 14,663*l*. and assets 8,464*l*. A composition of 10*s*. in the £ was accepted. What was the theoretical value of the composition?

11. A merchant mixes a lbs. of one kind of tea, b lbs. of a second, and c lbs. of a third, the cost prices of the three kinds being respectively p, q, r shillings per lb.; find the percentage of profit in his receipts when he sells at m shillings per lb.

12. A grocer mixes a lbs. of tea at p shillings per lb., b lbs. at q shillings per lb., c lbs. at r shillings per lb., and d lbs. at s shillings per lb. Required the retail price of the mixture in order that he may realize k per cent. profit on his outlay.

13. An apple-woman, having purchased l dozen of apples at p pence per dozen, m dozen at q pence per dozen, and n dozen at r pence per dozen, has to dispose of the three lots afterwards at $p+q+r$ pence per three dozen. Required, the condition that she should just realize her original outlay.

14. A fruit-dealer found that he had left on hand a quantity of peaches, of which he had bought one half at the rate of 5 for a shilling and the other half at the rate of 6 for a shilling, and, thinking to escape without loss, sold them all at the rate of 11 for 2 shillings, but found that by so doing he had incurred a loss of eighteenpence. At what rate ought he to have sold them in order to have made a profit of a guinea upon the transaction?

15. A grocer can sell coffee at 30 cents per lb., and realize a profit of 25 per cent. He, however, mixes the coffee with chicory, which cost him 6 cents per lb., and selling the mixture at 25 cents per lb. realizes a profit of 40 per cent. How much per cent. of coffee does the adulterated mixture contain?

16. How must a grocer mix tea at 2*s*. 6*d*. per lb. and 3*s*. per lb. in order to produce a mixture worth 2*s*. 8*d*. per lb.?

17. A sum of 23*l*. 14*s*. is to be divided between A, B, and C; if B gets 20 per cent. more than A, and 25 per cent. more than C, how much does each get?

SECTION V.—SIMPLE INTEREST.

ART. 28.—**Rate of Interest.** Interest is money paid for the loan of a sum of money. When the interest is made proportional to the sum lent and to the time during which it is lent, it is called simple interest. The rate of interest has then the form

r£ interest per £ principal per annum;

but as the interest is a quantity of a lower order than the principal, the rate is usually expressed in the percentage form, as for example—

5£ interest per 100£ principal per annum,

which is equivalent to

1/20 £ interest per £ principal per annum.

Expressed in the form of an equivalence, it is—

5£ interest = 100£ principal by year,

or 1£ interest = 20£ principal by year,

or 1£ interest per 20£ principal = year,

or 1£ interest per £ principal = 20 years,

or 1£ interest per year = 20£ principal.

When the dependent quantity in a rate and one of the independent quantities are expressed in terms of the same unit, the value of the rate is independent of the size of that unit. Thus, if "shilling" be substituted for "pound" in the rate of interest, the value of the rate remains unaltered. In such cases the size of the unit is indifferent, but its kind is determinate; in the case of interest it must be some unit of value.

ART. 29.—**Amount and Present Value.** Interest is an increment; by adding it to the principal we get the *amount*, that is, the new value of the principal. If, for one year,

r£ interest = £ principal,

then $1 + r$£ at end of year = £ at beginning. (1)

This is called the rate of improvement of money.

Let t denote any integral or fractional number, then

tr£ interest = £ principal, (2)

and $1 + tr$£ at end of t years = £ at beginning. (3)

The reciprocal of (1) is

$$\frac{1}{1+r}\text{£ at beginning of year} = \text{£ at end of year}, \qquad (4)$$

and this is the rate of *present value* for one year.

The reciprocal of (3) is

$$\frac{1}{1+tr}\text{£ at beginning of } t \text{ years} = \text{£ at end of } t \text{ years,} \qquad (5)$$

and this is the rate of present value for the term of t years.

By division we get

$$\frac{1}{1+tr} = 1 - tr + (tr)^2 - (tr)^3 + \text{etc.} \qquad (6)$$

When tr is small compared to 1, the next term $(tr)^2$ will be doubly smaller, and under that condition $1 - tr$ may be a sufficient approximation to the value of $\frac{1}{1+tr}$, and the rate of present value may be taken as

$$1 - tr\text{£ at beginning} = \text{£ at end of } t \text{ years.} \qquad (7)$$

ART. 30.—**Discount.** From (5) we deduce

$$1 - \frac{1}{1+tr}\text{£ difference} = \text{£ at end,}$$

i.e.,

$$\frac{tr}{1+tr} \quad \text{,,} \quad = \quad \text{,,} \qquad (8)$$

This is the *true* rate of discount, reckoning simple interest.

If the approximation in the previous article is justified, this expression reduces to tr. Then

$$tr\text{£ difference} = \text{£ at end.} \qquad (9)$$

This approximation is called Banker's Discount; it is commonly used in reckoning discount, excepting when the interval of time is long.

The value of the approximate rate of discount is the same as that of interest, only it is applied negatively, while the latter is applied positively.

A percentage is sometimes deducted from a charge for other reasons than the price of money; such a reduction is more properly termed an *abatement* than discount.

ART. 31.—**Equated Time.** Suppose that A owes B a£ due p months hence, b£ due q months hence, and c£ due r months hence;

it is required to find at what time the whole of the debts may be equitably discharged by the payment of an amount equal to the sum of the debts.

If the rate of interest is the same throughout the interval considered, the values of the several postponements are ap£ by month, bq£ by month, cr£ by month; hence the total value of the postponements is $(ap + bq + cr)$£ by month. The single sum to be paid is $(a + b + c)$£; and as the value of its postponement is to be equal to the values of the several postponements, the time for its payment will be

$$\frac{ap + bq + cr}{a + b + c} \text{ months.}$$

The quantity here considered would be more appropriately termed the *Equivalent* time. Analogous ideas occur in the subsequent chapters of this work.

EXAMPLES.

Ex. 1. How much per cent. per annum gives three farthings per half-a-crown per month?

$$\begin{aligned} \tfrac{3}{4} \text{ penny per } 30 \text{ pence} &= \text{month,} \\ \therefore\quad \tfrac{10}{4} \text{ pence per } 100 \text{ pence} &= \text{month,} \\ 12 \text{ months} &= \text{year;} \\ \therefore\quad \frac{10 \times 12}{4} \text{ pence per } 100 \text{ pence} &= \text{year,} \\ i.e.,\quad 30 \text{ pence per } 100 \text{ pence} &= \text{year,} \\ \text{or}\quad 30£ \text{ per } 100£ &= \text{year,} \\ \text{or}\quad 30 \text{ per cent. per year.} & \end{aligned}$$

Obsn. If it were stated that the interest was payable monthly in the one case, and yearly in the other, the transformation would involve compound interest.

Ex. 2. Find the simple interest on 321£ for $3\frac{1}{2}$ years at $4\frac{3}{4}$ per cent. per annum.

4·75£ interest = (100£) principal by year,

3·21 (100£) principal by 3·5 years;

$\therefore$ $3{\cdot}21 \times 3{\cdot}5 \times 4{\cdot}75£$ interest,

i.e., $53{\cdot}36625£$ interest

$$\begin{array}{r} 20 \\ \hline 7{\cdot}325 \\ 12 \\ \hline 3{\cdot}9 \end{array}$$

Answer.—53*l.* 7*s.* 3·9*d.*

Ex. 3. What principal will produce 37*l.* 5*s.* of simple interest in $2\frac{1}{2}$ years, at 3 per cent. per annum?

$3£$ interest per $100£$ principal $=$ year,

$2{\cdot}5$ years;

$\therefore$ $2{\cdot}5 \times 3£$ interest $= 100£$ principal,

$37{\cdot}25£$ interest;

$\therefore$ $\dfrac{3725}{2{\cdot}5 \times 3}£$ principal,

i.e., $\dfrac{1490}{3}£$ principal,

i.e., 496*l.* 13*s.* 4*d.*

Ex. 4. If 300\$ be laid out at simple interest for a certain number of years, it will amount to 360\$. If the same be allowed to remain two years longer, and at a rate of interest 1 per cent. higher, it will amount to 405\$. Find the rate and number of years.

$60\$ = 3 \times 100\$$ by x years;

$\therefore$ $\dfrac{20}{x}\$ = 100\$$ by year;

hence other rate of interest,

$$\frac{20}{x} + 1\$ = 100\$ \text{ by year,}$$

$3 \times 100\$$ by $x + 2$ years;

$\therefore$ $3\,(x + 2)\left(\dfrac{20}{x} + 1\right)\$$ interest.

Now this is given to be 105$ interest; hence the equation

$$(x + 2)\left(\frac{20}{x} + 1\right) = 35,$$

which reduces to

$$x^2 - 13x + 40 = 0,$$

the roots of which are 8 and 5. Hence 8 or 5 years, and $2\frac{1}{2}$ or 4 per cent.

Ex. 5. Calculate to the nearest halfpenny the true present value of a bill for 152*l.* 8*s.* payable on December 31, and discounted on July 17, at $3\frac{1}{2}$ per cent. interest per annum.

$\frac{7}{200}$£ interest = £ principal by year

$\frac{167}{365}$ year;

$$\therefore \quad \frac{167 \times 7}{200 \times 365}\text{£ interest} = \text{£ principal};$$

$$\therefore \quad \left(1 + \frac{167 \times 7}{200 \times 365}\right)\text{£ at end} = \text{£ at beginning},$$

$$152{\cdot}4\text{£ at end};$$

$$\therefore \quad \frac{152{\cdot}4}{1 + \frac{167 \times 7}{200 \times 365}}\text{£ at beginning},$$

$$\textit{i.e.,} \quad \frac{1524 \times 20 \times 365}{74169}\text{£ at beginning},$$

i.e., 149*l.* 19*s.* $11\frac{1}{2}$*d.*

Hence the true present value is 149*l.* 19*s.* $11\frac{1}{2}$*d.*

EXERCISE V.

1. What principal invested at 6 per cent. per annum is required to yield two bursaries, one of the yearly value of 35*l.*, the other of 17*l.* 10*s.*?

2. A person invests 3,000*l.*, part at 10 per cent. and the rest at 5 per cent. He gets an average of 7 per cent. for his money. How much is invested at each rate?

3. Find the amount of 560*l.* in $2\frac{1}{4}$ years at $4\frac{1}{2}$ per cent. simple interest.

4. In what time will 1,260*l.* amount to 1,496*l.* 5*s.* at $3\frac{3}{4}$ per cent. simple interest?

5. Calculate the interest on 1,529*l.* for 23 days at 5 per cent.

6. Find the interest on 375*l.* 16*s.* 8*d.* for 60 days at $4\frac{1}{4}$ per cent.

7. *A* lends *B* 2,500*l.* for 4 months. For how many months should *B* lend *A* 1,500*l.* in return, the rate of interest being one third higher?

8. At what rate per cent. per annum will 800*l.* become 9,500*l.* at simple interest in 30 years?

9. If 19*l.* 4*s.* amount to 22*l.* 16*s.* in three years at simple interest, what is the rate per cent. per annum?

10. A merchant bought 200 yards of cloth at 6*s.* per yard, payable in three months, and sold them one month after at 7*s.* per yard, payable in four months. To pay the purchase money he borrowed for the necessary time at the rate of 6 per cent. per annum. Find his gain or loss on the transaction.

11. A capitalist borrows 5,000*l.* in order to complete a sum of 19,000*l.* he is about to lend; he gains in one year's interest 795*l.* He borrows in another transaction, at a rate lower by one per cent. than in the previous one, a sum of 3,500*l.* to complete a loan of 14,000*l.*; he gains in this case 630*l.*, lending at the same rate as before. At what rates did he borrow and lend in the first transaction?

12. A man buys a site of 3 acres 3 roods 15 poles for a house and garden at 440*l.* an acre, and spends 3,808*l.* 15*s.* in building the house. He lets the premises for 357*l.* 10*s.* a year. What rate per cent. does he get for his money?

13. A deduction being made from a debt, 654*l.* 1*s.* is accepted in discharge of 686*l.* 13*s.* 4*d.* At what rate per cent. is the deduction made?

14. If a sum of 1,000*l.* becomes due four months hence, what is its present value as commonly calculated, and what as correctly calculated, interest being reckoned at 5 per cent.?

15. A merchant owes a sum of 2,183*l.* 15*s.* payable in four months from the present time. What sum ought he to pay down in order to satisfy the debt, reckoning interest at $4\frac{1}{2}$ per cent. per annum?

16. A sum of 1,928*l.* 17*s.* is due $1\frac{1}{4}$ years hence; what amount should be accepted for present payment, reckoning at $3\frac{1}{2}$ per cent.?

17. The rate of interest being 7 per cent., what is the true discount on a sum of 1,356*l.* 13*s.* 4*d.* due three months hence; and what is the interest on the same sum for nine months?

18. Find the true discount for 259 days on 100*l.* at $3\frac{1}{2}$ per cent. per annum.

19. Calculate the charge for the following:—

1 cwt. of indigo at 14*s.* 6*d.* per lb.
1 ton of cloves at 1*s.* 2*d.* per lb.
5 cwt. 3 qrs. 18 lbs. spelter at $4\frac{1}{2}$*d.* per lb.
7 cwt. 1 qr. 14 lbs. black tin at 64*l.* per ton.

Subtract 10 per cent. discount for cash.

20. Find the difference between the common and the true discount on a bill for 100*l.* due a year hence at 5 per cent.

21. Find the sum payable four months hence, interest being at 2 per cent., which will be equivalent to the two following sums:—422*l.* 12*s.* 6*d.* due three months hence at $2\frac{1}{2}$ per cent., and 485*l.* 10*s.* due five months hence at $2\frac{3}{4}$ per cent.; true discount being employed in all cases.

22. Goods were bought as follows on 1st April: 560*l.* on three months, 600*l.* on

4 months, 400*l.* on 5 months, and 1,120*l.* on 6 months; find the equated time of payment.

23. At what advance on cost must a merchant mark his goods, so that, after allowing 6 per cent. of his sales for bad debts, 7 per cent. of the cost for expenses, and an average credit of 6 months (money being worth 6 per cent.), he may make a clear gain of 15 per cent. on the first cost of the goods?

SECTION VI.—COMPOUND INTEREST AND ANNUITIES.

ART. 32.—**Rate of Interest.** Interest is said to be compound when the principal is allowed to grow by the continual addition of the interest at the end of a specified interval of time. Hence to the specification of the rate an additional specification is added—payable yearly, or half-yearly, as the case may be.

If the rate of interest is given in terms of the year, as, for example,

5£ interest per 100£ principal per year,

while the interval at which the interest is to be added to the principal is the half-year, we ought first to convert the rate into terms of the half-year by dividing by 2. In the above case,

2·5£ interest per 100£ principal per half-year,

or $\frac{1}{40}$£ interest per £ principal per half-year.

Suppose that we have reduced the rate of interest to the form

r£ interest per £ principal per interval,

where the interval referred to is the interval between the times at which the interest becomes due; then

$(1+r)$£ at beginning of 2nd interval = £ at beginning of 1st;

and $\frac{1}{1+r}$£ at beginning of 1st = £ at beginning of 2nd.

ART. 33.—**Future and Present Value.** Suppose that a sum of money is lent at compound interest, for the first interval at

p£ interest per £ principal per interval,

for the second at

q£ interest per £ principal per interval,

for the third at

r£ interest per £ principal per interval;

then we have

$(1+p)$£ at beginning of 2nd interval = £ at beginning of 1st,

$(1+q)$£ at beginning of 3rd interval = £ at beginning of 2nd,

$(1+r)$£ at beginning of 4th interval = £ at beginning of 3rd.

Hence by rule Art. 10,

$(1+p)(1+q)(1+r)$£ at beginning of 4th interval = £ at beginning of 1st,

or $(1+p)(1+q)(1+r)$£ at end of 3 intervals = £ at beginning of 3 intervals.

If the rates of interest are the same, that is, if $p=q=r$, the rate of growth or of *future value* becomes

$(1+r)^3$£ at end of 3 intervals = £ at beginning.

And generally for n intervals

$(1+r)^n$£ at end of n intervals = £ at beginning.

The reciprocal rate—the rate of present value—is

$\frac{1}{(1+r)^n}$£ at beginning of n intervals = £ at end.

When the period during which the interest accumulates comprises an integral number of intervals and a fraction of an interval, the calculation for the fraction of an interval is the same as for simple interest.

ART. 34.—**True Discount.** The rate of increment is—

$\{(1+r)^n-1\}$£ increment = £ at beginning,

or $\{(1+r)^n-1\}$£ increment = $(1+r)^n$£ at end.

The latter put into the form

$\left(1-\frac{1}{(1+r)^n}\right)$£ discount = £ debt due at end of n intervals

is the rate for calculating *true* discount when compound interest is reckoned.

Since $(1+r)^n = 1 + nr + \frac{n(n-1)}{1\cdot 2}r^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}r^3 +$, etc.,

if we take only the first two terms of this expansion, we get for the rate of discount

$$\left(1-\frac{1}{1+nr}\right)£ \text{ discount} = £ \text{ debt at end of } n \text{ intervals,}$$

which is the same as the rate derived from reckoning interest simply. (Art. 30.)

ART. 35.—**Annuity.** By an *Annuity* is meant a uniform payment made at equal intervals of time, generally a year or a half-year. It is specified in terms of £ paid per year, with an additional specification of payable yearly, or half-yearly, as the case may be. We shall suppose that it is payable yearly, but the same reasoning applies to any other interval of payment.

ART. 36.—**Amount of an Annuity.** To find the rate for the amount at the end of n years of an annuity payable yearly, and which has been paid at the end of each year in the period. By the previous Article,

$$\begin{aligned}
(1+r)^{n-1}£ \text{ at end of } n \text{ years} &= £ \text{ paid at end of 1st year,}\\
(1+r)^{n-2}£ \text{ at end of } n \text{ years} &= £ \text{ paid at end of 2nd year,}\\
(1+r)^{n-3}£ \text{ at end of } n \text{ years} &= £ \text{ paid at end of 3rd year,}\\
\cdots\quad\cdots &\quad\cdots\quad\cdots\\
(1+r)^{2}£ \text{ at end of } n \text{ years} &= £ \text{ paid at end of } (n-2)^{\text{th}} \text{ year,}\\
(1+r)£ \text{ at end of } n \text{ years} &= £ \text{ paid at end of } (n-1)^{\text{th}} \text{ year,}\\
1£ \text{ at end of } n \text{ years} &= £ \text{ paid at end of } n^{\text{th}} \text{ year;}
\end{aligned}$$

therefore

$$1+(1+r)+(1+r)^2+\ldots+(1+r)^{n-2}+(1+r)^{n-1}£ \text{ at end of } n \text{ years} = £ \text{ paid per year, for } n \text{ years.}$$

The above terms form a geometric progression, of which $(1+r)$ is the common ratio; hence their sum is

$$\frac{(1+r)^n-1}{r}.$$

Hence

$$\frac{(1+r)^n-1}{r}£ \text{ at end of } n \text{ years} = £ \text{ paid per year, for } n \text{ years.}$$

The reciprocal rate is

$$\frac{r}{(1+r)^n-1}\text{£ paid per year for } n \text{ years} = \text{£ at end of } n \text{ years};$$

from it we find what annuity paid for n years is equivalent to a given sum paid at the end of the n years.

ART. 37.—**Reversion of an Annuity.** Suppose that an annuity will begin to be reaped s years hence, and will continue to be reaped for n years. What is its present value? By the previous Article,

$$\frac{(1+r)^n-1}{r}\text{£ at end of period} = \text{£ per year.}$$

Now $$1\text{£ at present} = (1+r)^{n+s}\text{£ at end of period};$$

$$\therefore\ \frac{(1+r)^n-1}{r}\ \frac{1}{(1+r)^{n+s}}\text{£ at present} = \text{£ per year.}$$

When the present time is the beginning of the period of n years, $s=0$, and

$$\frac{1-\dfrac{1}{(1+r)^n}}{r}\text{£ at beginning of } n \text{ years} = \text{£ per year.}$$

ART. 38.—**Perpetual Annuity.** By a perpetual annuity is meant an annuity which is payable for an indefinitely great number of years, or half-years, as the case may be. We have seen that the value at the beginning of the period of payment is

$$\frac{1-\dfrac{1}{(1+r)^n}}{r}\text{£ at beginning} = \text{£ per year for } n \text{ years.}$$

When n, the number of years, is very great $1/(1+r)^n$ approximates to 0; hence, for a period containing a very large number of years of payment,

$$\frac{1}{r}\text{£ at beginning} = \text{£ per year for ever.}$$

Suppose that the annuity has been paid for a finite number of years, the value of the remainder when it begins to be paid is still

$$\frac{1}{r}\text{£ at present} = \text{£ per year for ever.}$$

The present value of any permanent property is connected with its annual value by the above rate. The value $1/r$ gives the number of *years' purchase.* It is considered as so many years, which, multiplying the annual income, gives the present worth of the property.

Similarly, according to the preceding article,

$$\frac{1-(1+r)^{-n}}{r}$$

is the number of years' purchase for a uniform annual payment extending over n years.

In the following table **G** denotes any unit of value. The table of entries, though short, is sufficient to give, with the aid of one multiplication, the entry for any year up to 59. For example, take 36 years. Multiply together the values for 30 and for 6 by the method of contracted multiplication. (See Example 2 following.)

RATE OF IMPROVEMENT OF MONEY AT COMPOUND INTEREST.

$(1+r)^n$ **G** at end of n years = **G** at beginning of n years.

n	3 per Cent.	4 per Cent.	5 per Cent.	6 per Cent.
	$\left(1+\frac{3}{100}\right)^n$	$\left(1+\frac{4}{100}\right)^n$	$\left(1+\frac{5}{100}\right)^n$	$\left(1+\frac{6}{100}\right)^n$
1	1·030000	1·040000	1·050000	1·060000
2	1·060900	1·081600	1·102500	1·123600
3	1·092727	1·124864	1·157625	1·191016
4	1·125509	1·169859	1·215506	1·262477
5	1·159274	1·216653	1·276282	1·338226
6	1·194052	1·265319	1·340096	1·418519
7	1·229874	1·315932	1·407100	1·503630
8	1·266770	1·368569	1·477455	1·593848
9	1·304773	1·423312	1·551328	1·689479
10	1·343916	1·480244	1·628895	1·790848
20	1·806111	2·191123	2·653298	3·207135
30	2·427262	3·243398	4·321942	5·743491
40	3·262038	4·801021	7·039989	10·285718
50	4·383906	7·106683	11·467400	18·420154

Given the above table, the calculation of elementary tables of the same kind for the other rates of this section is not laborious, provided a table of reciprocals is at hand. The exercise is all the more valuable because the results can be compared with the published tables.

Examples.

Ex. 1.—What is the equivalent of compound interest at $2\frac{1}{2}$ per cent. per quarter, payable each quarter, in terms of per cent. per annum, payable each year?

$1+\frac{1}{40}$£ at end of 1st quarter = £ at beginning,

$1+\frac{1}{40}$£ at end of 2nd quarter = £ at end of 1st,

$1+\frac{1}{40}$£ at end of 3rd quarter = £ at end of 2nd,

$1+\frac{1}{40}$£ at end of 4th quarter = £ at end of 3rd;

$\therefore$ $(1+\frac{1}{40})^4$£ at end of year = £ at beginning;

$\therefore$ $(1+\frac{1}{40})^4-1$£ interest = £ principal by year, payable yearly;

$\therefore$ $100\{(1+\frac{1}{40})^4-1\}$£ interest = 100£ principal by year.

Now, $$(1+\tfrac{1}{40})^4 = 1+4\tfrac{1}{40}+6\frac{1}{(40)^2}+4\frac{1}{(40)^3}+\frac{1}{(40)^4};$$

$$\therefore\quad 100(1+\tfrac{1}{40})^4 = 10+\tfrac{3}{8}+\tfrac{1}{160}+\tfrac{1}{25600}.$$

The first approximation to the value is 10, the second $10\frac{3}{8}$, the third $10\frac{61}{160}$, and the full value is $10\frac{9761}{25600}$. The first approximation is equivalent to reckoning simple interest.

Ex. 2.—Find the amount at compound interest, payable yearly, at the end of four years, of 2,345£, the rate of interest for the first year being 4 per cent., for the second 5 per cent., for the third 6 per cent., and for the fourth 7 per cent.

2,345£ at beginning,

$1+\frac{4}{100}$£ at end of 1st = 1£ at beginning,

$1+\frac{5}{100}$£ at end of 2nd = 1£ at end of 1st,

$1+\frac{6}{100}$£ at end of 3rd = 1£ at end of 2nd,

$1+\frac{7}{100}$£ at end of 4th = 1£ at end of 3rd;

hence, $2345\times 1{\cdot}04\times 1{\cdot}05\times 1{\cdot}06\times 1{\cdot}07$£ at end of 4th.

The calculation is as follows :—

$$\begin{array}{r} 2345 \\ 1{\cdot}04 \\ \hline 9380 \\ 2345 \\ \hline 2438{\cdot}80 \\ 1{\cdot}05 \\ \hline 1219400 \\ 243880 \\ \hline 2560{\cdot}7400 \\ 601 \\ \hline 25607400 \\ 1536444 \\ \hline 2714{\cdot}3844 \\ 701 \\ \hline 27143844 \\ 1900069 \\ \hline 2904{\cdot}3913 \end{array}$$

By decimalising we get 2,904*l.* 7*s.* 10*d.*

Three places of decimals in the answer are sufficient, for a farthing is very nearly the thousandth of a pound. Hence, when the decimals amount to 4, we begin to apply contracted multiplication. Put the highest figure of the multiplier under the lowest figure to be retained, reverse the order of the figures of the multiplier, and begin the multiplication by a figure at the figure of the multiplicand below which it falls. By this means we are able to cut off the unnecessary figures.

Ex. 3.—Find the number of years in which 1,000£ will amount to 2,400£ at 5 per cent. per annum, compound interest, payable yearly. Given $\log 3 = {\cdot}47712$, $\log 5 = {\cdot}69897$, and $\log 7 = {\cdot}84510$.

Suppose in n years, then

$$(1 + \tfrac{1}{20})^n\text{£ at end} = \text{£ at beginning},$$

$$1000\text{£ at beginning},$$

$$\therefore \quad 1000\,(1 + \tfrac{1}{20})^n\text{£ at end.}$$

But the amount at end is given to be 2400£ ;

$$\therefore \quad 1000(1 + \tfrac{1}{20})^n = 2400,$$

$$\therefore \quad (\tfrac{21}{20})^n = \tfrac{12}{5},$$

$$\therefore \; n\{\log 3 + \log 7 - \log 5 - 2\log 2\} = \log 3 + 2\log 2 - \log 5,$$

$$\therefore \quad n = \frac{\log 3 + 2\log 2 - \log 5}{\log 3 + \log 7 - \log 5 - 2\log 2}.$$

All the logs are given directly, excepting log 2.

$$\text{Now} \quad \log 10 = \log 2 + \log 5,$$

$$\therefore \quad \log 2 = 1 - \log 5 ;$$

$$\therefore \quad n = \frac{\log 3 - 3\log 5 + 2}{\log 3 + \log 7 + \log 5 - 2}.$$

$$\begin{array}{r} 2{\cdot}47712 \\ 2{\cdot}09691 \\ \hline {\cdot}38021 \end{array} \qquad \begin{array}{r} {\cdot}47712 \\ {\cdot}84510 \\ {\cdot}69897 \\ \hline {\cdot}02119 \end{array}$$

$$\therefore \qquad n = \frac{38021}{2119} = 18.$$

Ex. 4.—A tenant whose annual rent is 400£, due at the end of each year, wishes to pay the whole of his rent in advance at the beginning of a nineteen years' lease; what sum ought he to pay, interest being reckoned at 4 per cent. per annum?

400£ per annum,

$$\frac{(1+\frac{1}{25})^{19}-1}{\frac{1}{25}}\text{£ at end of 19 years} = \text{£ paid per annum},$$

$$1\text{£ at beginning of 19 years} = (1+\tfrac{1}{25})^{19}\text{£ at end};$$

$$\therefore \quad 400 \times 25\{1+\tfrac{1}{25})^{19}-1\}(1+\tfrac{1}{25})^{-19}\text{£ at beginning.}$$

$$i.e., \qquad 10000\{1-(\tfrac{25}{26})^{19}\}\text{£ at beginning.}$$

$$\log 25 = 1\cdot39794,$$

$$\log 26 = 1\cdot41497,$$

$$\therefore \qquad \log \tfrac{25}{26} = \bar{1}\cdot98297,$$

$$\text{and } 19 \log \tfrac{25}{26} = 19 \times \bar{1} + 19 \times \cdot98297,$$

$$= \bar{1}\cdot67643,$$

$$\therefore \qquad (\tfrac{25}{26})^{19} = \cdot47472.$$

Hence the sum is 5252£ +.

EXERCISE VI.

1. Find the simple and the compound interest on 378*l.* for three years at 5 per cent. per annum.

2. Find to a penny the compound interest on 377·57*l.* for 3 years at $3\frac{1}{2}$ per cent.

3. Which is greater, and by how much—the simple interest on 20*l.* for 3 years at 5 per cent., or the compound interest for the same time at $4\frac{1}{2}$ per cent.?

4. Find the amount, at compound interest, of 200*l.* for 3 years at 5 per cent. per annum.

5. A sum of money amounted to 465*l.* 2*s.* in 2 years at 5 per cent. compound interest; what was the sum?

6. Find the amount, at compound interest, at the end of 2 years, of 123*l.* at 4 per cent. per annum, payable half-yearly.

7. In how many years will a sum of money double itself at 5 per cent. compound interest?

8. A noble Scotch family have retained in their possession since the death of the Regent Murray in 1570, 500*l.* in gold coins. Find to what this sum would

have amounted by the present year (1884) if it had been invested at 5 per cent. compound interest.

9. Find to the nearest shilling the present value of 273*l.*, payable after 3 years, the rate of interest being 3 per cent. per annum.

10. Find the compound interest and true discount on 800*l.* for 3 years at 5 per cent.

11. The present value of a certain debt due 3 years hence is 112*l.* 10*s.* In 2 years' time, if it is not paid, its value will be 120*l.* 2*s.* 6*d.* What is the debt?

12. A gentleman leaves his property, worth 4000*l.*, to be divided between his sons, who are aged respectively 15 and 18 years at the date of his death. What sums ought his executors to set apart for the sons in order that they may receive the same amount on coming of age, taking the price of money at 5 per cent. per year?

13. A ship is valued at 14,720*l.* What sum should be insured at 8 per cent. by a person who owns one sixteenth of the ship, so that in case of loss he may recover both his share of the vessel and his insurance?

14. A gentleman insured his life for 250*l.* at a premium of 5*l.* per annum; he died after n years, and the insurance office neither gained nor lost in the transaction. Find n, reckoning compound interest at the rate of 5 per cent. per annum.

15. Find the present value of an annuity of 50*l.* payable for 12 years, *first,* when the annuity begins to be paid at the end of 1 year hence; and, *second,* when it begins to be paid at the end of 10 years hence. Money at 3 per cent. per annum.

16. Find the amount accumulated at the end of 3 years by a person who invests 500*l.* now, and does the same at the beginning of each succeeding year, at 8 per cent. compound interest.

17. A county borrows 150,000 dols., to be paid off, principal and interest, in 20 equal annual instalments. Find the annual payment, interest at 6 per cent.

18. A cargo of goods was insured at $3\frac{1}{8}$ per cent. on the price they were expected to fetch abroad, which was 20 per cent. over the price paid at home. The insurance came to 117*l.* 10*s.* Find the cost price of the goods.

19. How many years will it take 100*l.* to accumulate to 500*l.*, at 4 per cent. compound interest?

20. What is the rate of interest corresponding to 16 years' purchase?

21. The reversion of a freehold estate worth 200*l.* per annum to commence 4 years hence is to be sold. Ascertain its value at 5 per cent. compound interest.

22. Required the commutation for a perpetual pension of 4,000*l.* per annum, taking money at 4 per cent.

23. What is the number of years' purchase when 3 per cent. consols are bought at 96?

SECTION VII.—SHARES AND STOCKS.

ART. 39.—**Shares.** By *Shares* is meant the equal sums of money by which the capital of a public company is at first brought together. The original or nominal value of a share is the amount of money subscribed to the undertaking by a person for each share he receives; it is expressed in the form m£ nominal per share. Generally a portion of the value of a share is paid on application therefor, another portion on allotment, and the remainder may be called up at intervals as required by the state of the company. Hence besides the nominal value of a share we may have its paid-up value.

ART. 40.—**Premium, Par, Discount.** A person who has invested money in the shares of a company, may afterwards wish to exchange his shares for cash; the price received for a share will depend on the degree of prosperity of the company. Suppose that he sells at n£ per share. If n is greater than m, the shares are said to sell at a *premium*, the rate of premium being

$$(n-m)\text{£ premium per share.}$$

If n is less than m, the shares are said to sell at a *discount*, the rate of discount being

$$(m-n)\text{£ discount per share.}$$

If n is equal to m, the shares are said to sell at *par*.

ART. 41.—**Stock.** So long as the shares are not fully paid up, the capital of the company is sold only in shares; but when that has been done the shares are blended into one stock, and any integral number of pounds of the stock can be sold. The price of the capital of the company may then be no longer quoted at so much per share, but at so many £ cash for 100£ stock.

The portion of the profit accruing in the course of a year or a half year which it is thought safe to distribute is called the yearly or half-yearly *dividend*. It is divided among the shareholders in

proportion to the number of shares, or the amount of stock, held; hence we have rates of the form

d£ dividend per share,
d£ dividend per 100£ stock.

ART. 42.—**Government Stocks** or **The Funds.** When the Government borrows a sum of money temporarily, it gives notes to the persons who advance the money acknowledging the amount received, and agreeing to pay interest at so many pence per cent. per diem until the money is repaid. These notes are called Exchequer Bills, and they may pass from one person to another. Before the Revolution of 1688, it was customary to pay off temporary loans by raising a special fund; but since then it has frequently been the practice of the Government not to pay off the lenders, but to give them the right to a perpetual annuity equivalent to the sum advanced. This is now called funding the debt.

The annuity which is commonly offered is

3£ per annum per 100£ debt.

If, on converting a temporary loan into a permanent loan, 3£ per annum per 100£ debt is not considered a full equivalent, the sum lent is changed to a correspondingly-increased amount in the register of fund-holders. This changing of the amount of debt so as to keep the value of the annuity constant has been done with the view of facilitating the calculation of the sums due to the different fund-holders.

The amount of money upon which the annuities are reckoned is called Government Stock, the Funds, or the National Debt. It consists of several classes of stock, as the "3 per cent. Consols," so called because various stocks were consolidated into one uniform stock; the "3 per cent. Reduceds," so called because several stocks paying higher annuities were reduced to 3 per cent., etc. The annuities are payable half-yearly, and the total sum paid is spoken of as the half-yearly dividend.

The Government reserve the right to pay off any amount of stock by an equal amount of cash; but a fund-holder cannot, on his own motion, get cash for his stock unless by selling it to a third party. Government stock is sold at the rate of so many £ cash for 100£ stock.

ART. 43.—**Brokerage.** The purchase or sale of stock is usually made through a middleman called a stock-broker. The commission which he charges is called *brokerage;* it is usually at the rate of $\frac{1}{8}$£ per 100£ stock bought or sold.
Brokerage is charged both from the seller and from the buyer, so that the gain to the stockbroker, provided the two prices are the same, is at the rate of

$\frac{1}{4}$£ per 100£ stock bought and sold.

ART. 44.—**Equivalent Amounts of Stock.** Let us consider two kinds of stock, distinguishing them as A and B. Let their rates of dividend be

$$3£ \text{ dividend per year} = 100£ \text{ of } A \text{ stock},$$
$$4£ \text{ dividend per year} = 100£ \text{ of } B \text{ stock};$$

the reciprocals of these rates are

$$\tfrac{1}{3} \times 100£ \text{ of } A = £ \text{ per year},$$

and $$\tfrac{1}{4} \times 100£ \text{ of } B = £ \text{ per year};$$

hence $$\tfrac{1}{4}£ \text{ of } B = \tfrac{1}{3}£ \text{ of } A,$$

or $$3£ \text{ of } B = 4£ \text{ of } A,$$

or $$75£ \text{ of } B = 100£ \text{ of } A.$$

Here the equivalence is in respect of producing the same amount of income. Thus, having given a number of stocks, we can select a given amount (say £100) of one of them, and find the amount of each of the others which produces an equal amount of income.

The price of a stock (other things being equal) is in proportion to the income which it brings in. Hence 75£ of B will fetch the same amount of cash as 100£ of A; and the rate connecting

the prices which will fetch equal amounts of the two stocks will be

100£ cash for B = 75£ cash for A.

This rate of equivalent prices is the reciprocal of the previous rate of equivalent stocks.

Examples.

Ex. 1. A man buys 650*l.* stock in the 3 per cents. at 90, and invests £400 in the 4 per cents. at 110; find the whole sum expended, and the annual income secured.

90£ paid = 100£ stock,
650£ stock;
∴ 65 × 9£ paid, *i.e.*, 585£.

Hence the whole sum expended is 985£.

Again, 3£ per year = 100£ stock,
650£ stock;
∴ 3 × 6·5£ per year.

Again, 4£ per year = 100£ stock,
100£ stock = 110£ invested,
400£ invested;

$$\therefore \quad \frac{4 \times 400}{110}\text{£ per year,}$$

i.e., $14\cdot\dot{5}\dot{4}$£ per year.

Hence the total income is 34·045£ per year,
i.e., £34 0*s.* 11*d.* per year.

Ex. 2. A person invests 1,271£ in the 3 per cents. at 93; how much will he gain by selling out at $94\frac{1}{8}$?

93£ paid = 100£ stock,
$94\frac{1}{8}$£ received = 100£ stock;
∴ $94\frac{1}{8}$£ received = 93£ paid,
∴ $1\frac{1}{8}$£ profit = 93£ paid,
1,271£ paid;

$$\therefore \quad \frac{1,271 \times 9}{93 \times 8}\text{£ profit,}$$

$$i.e., \quad \frac{123}{8}\text{£ profit,}$$

$$i.e., \quad \text{£15 } 7s.\ 6d. \text{ profit.}$$

Ex. 3. Which is the better investment—bank stock paying 10 per cent. at 319, or 3 per cent. consols at 96 ?

In the first case,

$$10\text{£ per year} = 319\text{£ cash,}$$

$$i.e., \quad 1/31{\cdot}9\text{£ per year} = \text{£ cash.}$$

In the second case,

$$3\text{£ per year} = 96\text{£ cash,}$$

$$\therefore \quad 1/32 \text{ per year} = \text{£ cash.}$$

Hence the former is better by

$$\left(\frac{1}{31{\cdot}9} - \frac{1}{32}\right)\text{£ per year} = \text{£ cash,}$$

which is about $\frac{1}{100}$£ per year $= 100$£ cash.

Ex. 4. A person has \$20,000 invested in stock paying 6 per cent. which he sells and invests in stock paying 7 per cent. at $87\frac{1}{2}$. If the increase in his income be \$40, what is the price of the first-named stock ?

$$\$20{,}000 \text{ stock,}$$

$$6\$ \text{ income} = 100\$ \text{ stock;}$$

$$\therefore \quad 1{,}200\$ \text{ income.}$$

Again, $\quad 20{,}000\$$ 1st stock,

$$x\,\$ \text{ cash} = 100\$ \text{ 1st stock,}$$

$$100\$ \text{ 2nd stock} = 87{\cdot}5\$ \text{ cash,}$$

$$7\$ \text{ income} = 100\$ \text{ 2nd stock;}$$

$$\therefore \quad \frac{x200 \times 7}{87{\cdot}5}\$ \text{ income,}$$

$$i.e., \quad 16x\,\$ \text{ income.}$$

Now, it is given that

$$16x - 1200 = 40;$$

$$\therefore \quad x = 77\tfrac{1}{2}.$$

Ex. 5. A, B, and C went into a joint-stock business, in which A invested 50*l.*; B, 60*l.*; and C, 80£. B withdrew his money after 10 months, but A and C continued in the business for 6 months longer and then wound it up. The total profit was found to be 335£. How much should each receive?

A,	800£ by month,
B,	600£ by month,
C,	1,280£ by month,
Total,	2,680£ by month.

Hence 335£ dividend = 2,680£ by month,

i.e., 1£ dividend = 8£ by month.

Hence A gets 100£ dividend,

B gets 75£ dividend,

C gets 160£ dividend.

EXERCISE VII.

1. How much 4 per cent. stock at 96 can be bought for 1,000*l.*? What annual income will it produce?

2. Find the income of a person who invests 10,098*l.* in the 3 per cents. at $93\frac{5}{8}$, paying his broker $\frac{1}{8}$ per cent.

3. What sum must be invested in the $3\frac{1}{2}$ per cents. at 104 to obtain an income of 329*l.*?

4. A person invests 1,000*l.* in the stocks at $92\frac{1}{2}$. At what price must he sell out to clear 50*l.*?

5. A person buys 1,000*l.* 3 per cent. stock at $96\frac{1}{4}$ and sells out at $88\frac{1}{2}$. How much does he lose thereby? If he reinvests his money at 4 per cent., find the alteration in his income.

6. I sell out 5,000*l.* four per cent. stock at 108, and with the proceeds buy five per cents. at 120; what is the change in my income?

7. What must be the price of a 6 per cent. stock in order that money invested in it may yield $4\frac{1}{2}$ per cent.?

8. If a person were to transfer 29,000*l.* stock from the $3\frac{1}{2}$ per cents. at 99 to the 3 per cents. at $90\frac{5}{8}$, what would be the difference in his income?

9. A proprietor of 3 per cent. consols receives his half-yearly dividend and lays it out in the purchase of more consols at 90. His next half-year's dividend is 457*l.* 10*s.* By how much does this dividend exceed the former?

10. What must be the price of consols in order that, after deducting income-tax of 5*d.* in the £, there may remain a clear return of $3\frac{1}{4}$ per cent. on the cost?

11. A stock is paying 5 per cent. dividend, but after seven years the dividend is to be reduced to 4 per cent. What ought to be paid per 100*l*. share, the value of money being 4 per cent. ?

12. A person invests 5,445\$ in stock paying 6 per cent. when at $90\frac{3}{4}$, and on the stock rising to 91 transfers to another stock paying 7 per cent., which is selling at $97\frac{1}{2}$; how much is his income increased?

13. A man invests 3,600*l*. in 3 per cent. stock at 90. He sells out at 80, and lends $\frac{3}{8}$ths of his money at 4 per cent., and $\frac{5}{8}$ths at 5 per cent. How long must the loan last, so that when he re-invests his money in 3 per cents. at 90, his gain on interest (simple) may exactly equal his loss upon principal?

14. A person has an income derived from 3,360*l*. which was originally invested in the four per Cents. at 96. If he now sells out at 94, and invests one half of the proceeds in railway stock at $82\frac{1}{4}$, which pays a dividend of 3 per cent., and the other half in bank stock at $164\frac{1}{2}$, paying $8\frac{1}{2}$ per cent. dividend, what difference will he find in his income?

15. A certain 3 per cent. stock is at $91\frac{1}{2}$, and a 4 per cent. stock at 123. One person buys 1,000*l*. stock in each, and another person invests 1,000*l*. in each; compare the respective rates of interest obtained by the two on their whole investments.

16. The capital of a trading company consists of 4,000 *A* shares of 80*l*. each and 2,000 *B* shares of 25*l*. each; in dividing the profits 5 per cent. of the amount of each share is first paid, and then the remainder, if any, is divided equally amongst the shareholders. The profits of the undertaking in one year were 34,853*l*. 12*s*. $6\frac{1}{2}$*d*. How much would be paid to the holder of an *A* share and how much to the holder of a *B* share?

17. *A* buys 100*l*. of 3 per cent. stock at 90, and *B* buys 100*l*. of $4\frac{1}{2}$ per cent. stock at 135; what return per cent. will each get for his money?

18. A man has 1,583*l*. 17*s*. 11*d*. in 3 per cent. stock, and 982*l*. 12*s*. 6*d*. in $3\frac{1}{2}$ per cent. stock; he transfers a certain sum from the former to the latter when the stocks are at 91 and 98 respectively, and thus makes the income derived from each the same. How much has he finally in 3 per cent. stock?

19. A dividend is announced at the rate of $6\frac{1}{2}$ per cent., free of income-tax. What is the rate, not free of income-tax, when the rate of the tax is fivepence per £?

20. The Chancellor of the Exchequer proposes to reduce the 3 per cents. to $2\frac{3}{4}$ per cents., giving the fund-holder the option of being paid at par or of receiving 102*l*. of the new stock for every 100*l*. of the old. What percentage of gain or loss in his income does a fund-holder sustain who chooses the latter alternative?

21. It is proposed to issue a new $2\frac{1}{2}$ per cent. stock at 108*l*. Find the percentage return to be obtained by investing in it.

22. A man invests 1,000*l*. equally in shares of two banks. The shares of the one are at 3 per cent. discount, and of the other at 5 per cent. premium; the price of stock in the former suddenly rises 7 per cent., and that in the latter falls

6 per cent. lower than when the purchase was made. If the man now sells out what will he gain or lose?

23. The expense of constructing a railway is 1,000,000*l*., of which 40 per cent. is borrowed on mortgage at 6 per cent. and the remainder is held in shares; what must be the average weekly receipts so as to pay the shareholders 5 per cent., the working expenses being 65 per cent. of the gross receipts?

24. A company with a capital of 200,000*l*. paid 7 per. cent. to the shareholders; afterwards a new issue of stock was ordered, and the profit to be divided became six times as much as at first, yet the company could pay only 3 per cent. dividends; find the amount of new stock issued.

25. The total amount of the three categories of three per cent. stock is 612,000,000*l*. By how much would the National Debt be increased by the conversion of the whole of this stock according to the proposal of Question 20?

SECTION VIII.—EXCHANGE.

ART. 45.—**Foreign Money.** The units of value in use in the principal countries of the world are given in the accompanying table, together with their approximate equivalents in sterling money. It will be observed that several Monetary Unions have been formed.

ART. 46.—**The Latin Union.** The Latin Union was instituted by France, Belgium, Switzerland, and Italy in 1865; since then it has been joined by Greece, Spain, Roumania, and Servia. The units are the *franc* and the *centime*, which is the one hundredth part of the franc; there is also the *decime*, the tenth part, but it is not much used. These units have been adopted by all the countries mentioned, but in some cases under different names. Some of the latter countries have not yet adopted the standard of the Union.

The standard is gold; and the standard fineness is 900 parts by weight of pure gold in 1,000 parts by weight of the alloy. The

gold coins are the 20-franc piece called a Napoleon, 100-franc piece, 50-franc piece, 10-franc piece, and 5-franc piece. The weight of the 20-franc piece is 6·45161 grammes, and that of any of the others is in proportion to its indicated value.

The silver coins are 5 francs, 2 francs, 1 franc, ½ franc, ⅕ franc. The millesimal fineness of the 5-franc piece is 900; that of the others is 835. A silver 5-franc piece weighs 25 grammes; it is at present legal tender for any amount, thus making the standard in a manner double. The other silver coins are only token money.

The bronze coins are—

10 centimes, having a weight of 10 grammes and diameter of 30 millimetres.

5 centimes, having a weight of 5 grammes and diameter of 25 millimetres.

2 centimes, having a weight of 2 grammes and diameter of 20 millimetres.

1 centime, having a weight of 1 gramme and diameter of 15 millimetres.

ART. 47.—**United States.** The units of value in American States were founded on the old Spanish silver dollar, the value of which fluctuates between 3*s.* 6*d.* and 4*s.* 6*d.* sterling, according to the price of silver.

In the United States the standard is practically the single gold standard. The fundamental unit of value is denominated the *eagle.* The eagle is 258 grains in weight, and its fineness is 900 grains of pure gold in 1,000 grains of the coin.

The unit of account is the dollar, defined as the tenth part of the eagle. It is equivalent to 10 dimes, or 100 cents, or 1,000 mills; but only dollars and cents are used in accounts.

The gold coins are (besides the eagle) double-eagle, half-eagle, 3-dollar, 2½-dollar, dollar, all of the fineness of the eagle, and having a weight in proportion to their indicated value.

The silver coins are dollar, half-dollar, quarter-dollar, 10-cents, 5-cents.

There is a nickel 5-cents and a copper-nickel cent.

FOREIGN AND COLONIAL UNITS OF VALUE.

		Approx. equiv. s. d.
Latin Union—		
France, Belgium, Switzerland,	1 franc = 100 centimes,	
Italy,	1 lira = 100 centesimi,	0 $9\frac{1}{2}$
Spain,	1 peseta = 100 centesimos,	
Greece,	1 drachme = 100 lepta,	
Roumania,	1 lëu = 100 bans,	
Servia,	1 dinar = 100 paras,	
Scandinavian Union—		
Denmark, Sweden, Norway,	1 krone = 100 örer,	1 $1\frac{1}{3}$
Germany,	1 mark = 10 groschen = 100 pfennige,	0 $11\frac{3}{4}$
Austria,	1 gulden = 100 kreutzer,	1 $11\frac{1}{2}$
Netherlands,	1 guilder = 100 cents = 20 stivers,	1 8
Russia,	1 rouble = 100 copecks,	3 2
Turkey,	1 medjidie = 100 piastres,	18 0
Portugal,	1 milrei = 1000 reis,	4 5
Brazil,	1 milrei = 1000 reis,	2 $2\frac{1}{2}$
Dollar Countries—		
United States,	1 dollar = 100 cents,	4 $1\frac{1}{2}$
Canada,	1 dollar = 100 cents,	4 $1\frac{1}{2}$
Mexico,	1 dollar = 100 cents,	4 $0\frac{1}{2}$
Chili,	1 peso = 100 centavos,	3 9
Peru,	1 sol = 100 centesimos,	3 $11\frac{1}{2}$
Japan,	1 yen = 100 sen,	4 1
India,	1 rupee = 16 annas = 64 pice = 192 pie,	1 $10\frac{1}{2}$
China,	1 tael = 10 mace = 100 candareens = 1,000 cash,	5 2
Australia, South Africa, British West Indies,	1 pound sterling = 20 shillings = 240 pence.	

"Milliard" = 1,000 million francs. "Lac" = 100,000 rupees.
"Crore" = 10,000,000 rupees. "Conto" = 1,000 milreis.

ART. 48.—**Germany.** One system extends over the different States of the German Empire. It was established in 1876. The

standard is gold; 900 parts fine in 1,000 parts of alloy. The cold coins are 20-mark pieces, 10-mark pieces, 5-mark pieces. The weight of the 10-mark piece is 3·982477 grammes; the others have weights in proportion to their values. The silver coins are 5-mark, 2-mark, 1-mark, 50-pfennige, 20-pfennige. Their millesimal fineness is 900, and they are legal tender up to 20 marks. The "thaler" silver pieces have not yet been demonetized; they continue to be legal tender for three marks.

ART. 49.—**The Scandinavian Union.** The standard of the Scandinavian Union is also gold. The gold coins are 10-kroner and 20-kroner, having a millesimal fineness of 900. The weight of the former coin is 4·480286 grammes. The krone itself is represented by a silver coin which is legal tender up to 20 kroner only. This Monetary Union was instituted in 1875.

ART. 50.—**Par of Exchange.** By the par of exchange is meant the rate connecting the unit of value of one country with the unit of value of another country, the intrinsic value only of the units being considered. When both units are defined in terms of gold of a constant fineness, the value of the rate is invariable; but when one of the units is defined in terms of gold, and the other in terms of silver, the value of the rate fluctuates with the price of silver.

Example of the first case: To find the par of exchange between the *pound* and the *franc.*

$$
\begin{aligned}
1 \text{ pound} &= 123{\cdot}274 \text{ grains of standard gold},\\
12 \text{ grains of standard gold} &= 11 \text{ grains of pure gold},\\
15{\cdot}432 \text{ grains} &= 1 \text{ gramme},\\
9 \text{ grammes of pure gold} &= 10 \text{ grammes of standard gold},\\
1{,}000 \text{ grammes of standard gold} &= 3{,}100 \text{ francs};\\
\therefore \quad 12 \times 15{\cdot}432 \times 9 \text{ pounds} &= 123{\cdot}274 \times 11 \times 31 \text{ francs},
\end{aligned}
$$

which, when reduced, becomes

$$25{\cdot}222 \text{ francs} = \text{pound}.$$

Observe—The third equivalence is true universally, but all that is required for the reasoning is that

15·432 grains of pure gold = 1 gramme of pure gold.

Example of the second case: To find the par of exchange between the *pound* and the *rupee*, when silver is at 5*s*. per oz. of standard fineness (37 parts pure in 40 of standard).

A rupee contains $\frac{3}{8}$ oz. of silver, $\frac{11}{12}$ fine.

$$\begin{aligned} 1 \text{ rupee} &= \tfrac{3}{8} \text{ oz. of standard silver,} \\ 12 \text{ oz. of standard silver} &= 11 \text{ oz. of pure silver,} \\ 37 \text{ oz. of pure silver} &= 40 \text{ oz. of standard silver,} \\ 1 \text{ oz. of standard silver} &= 60d.\,; \\ \therefore \quad 12 \times 37 \times 8 \text{ rupees} &= 3 \times 11 \times 40 \times 60 \text{ pence,} \end{aligned}$$

which reduces to

22·3 pence = rupee.

PARS OF EXCHANGE.

(Extracted from Tate's *Modern Cambist*).

BOTH STANDARDS GOLD.

Commercial Centres.	Par of Exchange.	Common Value.
Paris and London,	25·2215 francs = £	25·22
Amsterdam ,,	12·1071 florins = £	12
Berlin ,,	20·42945 marks = £	$20{\cdot}42\frac{1}{2}$
New York ,,	4·866564 dols. = £	$4{\cdot}86\frac{2}{3}$
Lisbon ,,	53·285 pence = milreis	$53\frac{3}{10}$
Copenhagen ,,	18·15952 kroner = £	$18\frac{1}{4}$
Vienna ,,	10·215 florins = £	
St. Petersburg ,,	38·177 pence = rouble	
Montreal ,,		$4{\cdot}86\frac{2}{3}$

ONE STANDARD SILVER.

Price of silver taken at 5*s*. per oz. of standard fineness.

Calcutta and London,	22·3 pence = rupee
Shanghai ,,	70·95 pence = tael.

It will be observed that the rate it sometimes expressed in the reciprocal form. The reciprocal is in these cases the customary form. The independent unit is called the *fixed* price, and the dependent quantity the *variable* price.

ART. 51.—**Deduction of Approximate Rates.** We have found

that	25·22 francs	=	1 pound,
or	2522 francs	=	100 pounds,
or, dividing by 2,	1261 francs	=	50 pounds.

As the numbers 1261 and 50 are prime to one another, it is impossible to simplify the fraction further. But a series of approximations can be derived, and the process for deriving them is as follows :—

```
50)1261(25
    100
    ---
     261
     250
     ---
      11)50(4
         44
         --
          6)11(1
             6
            --
             5)6(1
               5
               -
               1)5(5.
```

The greater number is divided by the less, the less by the remainder, and so on until there is no remainder. From the division we derive

$$\frac{1261}{50} = 25 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{5}}}}$$

The first approximation is 25,

The second approximation is $25 + \frac{1}{4} = 101/4$,

The third approximation is $25 + \cfrac{1}{4 + \cfrac{1}{1}} = 126/5$,

The fourth approximation is $25 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1}}} = 227/9$,

the fifth being the full value. Hence the approximate pars, beginning with the roughest, are

$$\begin{aligned} 25 \text{ francs} &= 1 \text{ pound}, \\ 101 \text{ francs} &= 4 \text{ pounds}, \\ 126 \text{ francs} &= 5 \text{ pounds}, \\ 227 \text{ francs} &= 9 \text{ pounds}. \end{aligned}$$

The above process is called the process of continued fractions, and the several approximations to the given fraction are said to be convergents to it.

ART. 52.—**Course of Exchange.** By the course of exchange is meant the rate connecting the units of value of two countries, according to which bills of exchange for the time being are negotiated. A bill of exchange is a written order for the payment of a specified sum of money. The credit and debit accounts of two countries can be settled by means of bills, without the transference of coin, provided the credit and debit are equal. In general there will be a difference, and the existence of this difference affects slightly the rate of exchange. In the country to which the balance is favourable, the exchange for its unit of value will be somewhat higher than the par amount; in the country to which the balance is unfavourable, the exchange for its unit of value will be slightly less than the par amount. This is owing to the relative abundance or scarcity of bills of exchange. The course of exchange never exceeds the par of exchange by more than the cost of sending coin or bullion.

When the bill is not payable until the expiry of some months, the course of exchange is further affected by the deduction of interest due to deferred payment. When the course of exchange involves interest it is said to be long; when it does not involve interest it is said to be short.

Suppose that the course of exchange between Paris and London is

$$25{\cdot}50 \text{ franc} = £.$$

Since the par of exchange is

$$25{\cdot}22 \text{ franc} = £,$$

we deduce $25{\cdot}50 - 25{\cdot}22$, *i.e.*, $\cdot 28$ franc premium $= £$.

If the course of exchange is less than the par, for example,

$$25{\cdot}15 \text{ franc} = £;$$

then

$$\cdot 07 \text{ franc discount} = £.$$

When the course of exchange is equal to the par of exchange, the exchange is said to be at par.

ART. 53.—**Indirect or Arbitrated Rates of Exchange.** The indirect rate of exchange between two places is the rate deduced from the direct rate of each with respect to a third place. There may be more than one intermediate place.

For example.—The direct rate of exchange between London and Berlin is $20{\cdot}66$ marks $= £$, and the direct rate between Berlin and Paris is 81 marks $= 100$ francs; deduce the indirect rate of exchange between Paris and London,

$$20{\cdot}66 \text{ marks} = £,$$
$$100 \text{ francs} = 81 \text{ merks};$$
$$\therefore \quad 2066 \text{ francs} = £,$$
$$\textit{i.e.,} \quad 25{\cdot}50\tfrac{1}{2} \text{ francs} = £.$$

EXAMPLES.

Ex. 1. Convert 10,000 francs into sterling money, when exchange is at 25 francs 20 centimes per £ sterling.

$$25 \cdot 20 \text{ francs} = £,$$
$$10{,}000 \text{ francs};$$
$$\therefore \quad \frac{10{,}000}{25 \cdot 20} £,$$
$$i.e., \quad 396£ \ 16s. \ 6d.$$

Ex. 2. Express 1,000 dollars in francs; given that 1 dollar is equivalent to 4*s.* $1\frac{1}{2}d.$, and one franc to $9\frac{1}{2}d.$

$$1{,}000 \text{ dollars},$$
$$49 \cdot 5 \text{ pence} = 1 \text{ dollar},$$
$$1 \text{ franc} = 9 \cdot 5 \text{ penny};$$
$$\therefore \quad \frac{1{,}000 \times 495}{95} \text{ francs},$$
$$i.e., \quad \frac{99{,}000}{19} \quad ,,$$
$$i.e., \quad 5{,}210 \text{ francs } 53 \text{ centimes}.$$

Ex. 3. The fineness of English silver coin is 37 parts by weight of silver to 3 parts by weight of alloy; express its fineness in millièmes.

Let **M** denote any unit of weight (see Art. 54), then

$$37 \ \mathbf{M} \text{ silver} + 3 \ \mathbf{M} \text{ alloy} = 40 \ \mathbf{M} \text{ standard};$$
$$\therefore \quad \tfrac{37}{40} \ \mathbf{M} \text{ silver} = 1 \ \mathbf{M} \text{ standard};$$
$$\therefore \quad \frac{37 \times 100}{4} \ \mathbf{M} \text{ silver} = 1000 \ \mathbf{M} \text{ standard},$$
$$i.e., \quad 925 \quad ,, \quad = 1000 \quad ,,$$

EXERCISE VIII.

1. A German 20-mark piece is worth ·979*l.*; find to the nearest farthing the value in English money of 3725·39 marks.

2. Given that the United States dollar is equivalent to 4*s.* $1\frac{1}{2}d.$; find the equivalent in American money of the several current British coins.

3. Reduce 40,000 dollars to pounds sterling and to francs, taking par of exchange.

4. From the table of pars of exchange deduce the par of exchange between Paris and New York, and between Berlin and Paris.

5. Exchange 78*l.* 6*s.* 8*d.* into francs at $25 \cdot 24\frac{1}{2}$.

6. Exchange a milliard into pounds sterling at 25·22.

7. Exchange 1,000 marks into pounds sterling at 20·45 marks per £.

8. Exchange 1,000 milreis into sterling money at 52*d.* per milreis.

9. Exchange £100 into roubles at 26*d.* per rouble.

10. Exchange 10,000 dollars into pounds at $48\frac{5}{8}$ dollars per £10.

11. The direct exchange between St. Petersburgh and Berlin is 215 marks = 100 roubles, and that between Berlin and Vienna is 175 marks = 100 florins; what is the arbitrated rate between St. Petersburgh and Vienna?

12. A money-changer buys francs for $9\frac{1}{2}d.$ each, and sells them at the rate of 25 for a sovereign. How much money must pass through his hands in this way in order that he may gain £150?

13. A Canadian Company borrows in Paris 294,000 francs for which it pays an annual interest of 2,920 dollars. This loan is transmitted through London when exchange in London is quoted at 25·30 francs, and sterling exchange is $109\frac{5}{8}$. Find what rate of interest the Company pays on the money actually received.

14. A quantity of sugar, valued at 42,134 dollars Spanish gold, was entered for duty at 30 per cent. In consequence of Spanish gold having been taken at par, whereas it was only worth $92\frac{1}{2}$ cents on the dollar, a refund of duty was afterwards claimed. Calculate the amount.

15. The full weight of the sovereign is 123·274 grains; its fineness is 22 carats. What is the weight of gold in a sovereign?

16. The value of an ounce of the gold of which sovereigns are made is 3*l.* 17*s.* $10\frac{1}{2}$d. What is the weight in pounds troy of 46,725 sovereigns?

17. How many "Napoleons" are required to weigh a kilogramme?

18. At New York a bill of exchange on Dublin for £720 cost 3,472 dollars; find the course of exchange.

19. A Glasgow merchant ships to his Montreal agents for sale goods for which he pays £616 sterling in Glasgow. He pays an *ad valorem* duty of 12 per cent. upon the goods, and a commission of 7 per cent. to his agents for their services. The goods realize in Montreal 7,800 dollars. Find the merchant's net gain, a pound sterling being equal to 4·86 dollars.

CHAPTER SECOND.

GEOMETRICAL.

SECTION IX.—LENGTH.

ART. 54.—**General Unit.** The idea of length is one of the fundamental ideas of exact science. We shall denote, following Clerk-Maxwell in the choice of a letter,[1] any unit of length by the capital letter **L**. It denotes any unit of length in the same way as n denotes any number. We have chosen a bold and simple form of the letter in order that there may be a clear contrast between it and any italic letter used to denote a numerical value.

ART. 55.—**Imperial Standard of Length.** In the Imperial System of Weights and Measures[2] the standard unit of length is determined by means of a bronze bar, constructed in 1845 and now deposited in the Standards Department of the Board of Trade in the custody of the Warden of the Standards. In a small hole near each end of the bar there is a gold plug, and across each plug there are drawn three transverse lines. The distance between the centres of the middle transverse lines, when the bar is at the temperature of 62° Fahrenheit, is the standard *yard*. The previous standard yard, constructed in 1760, was lost in the fire which destroyed the Houses of Parliament in 1834. It had been defined as bearing at 62° Fahr. the proportion of 36

[1] *Electricity and Magnetism*, vol. 1, p. 3.

[2] Weights and Measures Act, 1878.

inches to 39·1393 inches, the length of a pendulum vibrating seconds in a vacuum in the latitude of London and at the level of the sea. On the recommendation of a scientific commission, this provision for its restoration was repealed, and a new standard constructed from authentic copies. To provide for the restoration of the new standard, should it be destroyed, four copies (called parliamentary copies) were constructed, and distributed to the Royal Mint, the Royal Society of London, the Royal Observatory at Greenwich, and the New Palace at Westminster. The recent Act provides for the construction of a fifth parliamentary copy. From these copies are derived the Board of Trade standards, and from the latter the local standards for testing measures used in trade.

The oldest standard yard now existing is the exchequer yard of Henry VII. It falls short of the existing standard by only the one hundredth part of an inch. It is represented in the accompanying illustrations. In form it is an end-measure, the present standard being a line-measure.

Exchequer yard of Henry VII., left-hand end, showing sixteenths of yard.

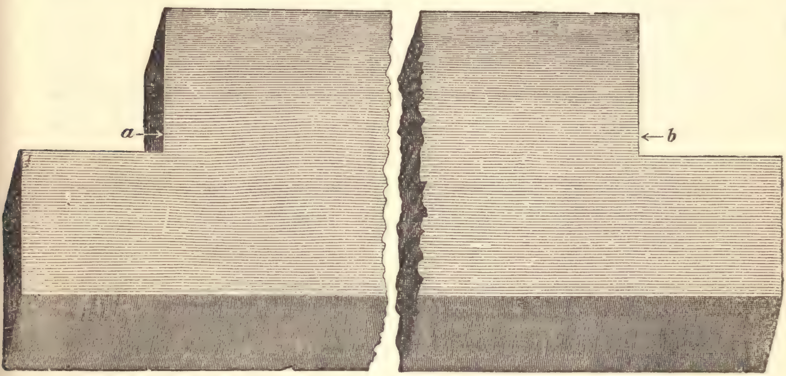
Right-hand end, showing inches.

ART. 56.—**Derived Units of Length.** The other units of length in the Imperial system having a special name, such as the *inch*, the *foot*, the *mile*, are defined as a multiple or submultiple of the yard. Originally, no doubt, they were defined independently of one another, but the definitions were rough. The "foot" meant the length of a man's foot, the "yard" the length of a man's arm, and the "inch" the breadth of a man's thumb; and three feet were equivalent to one yard, and 12 inches to one foot, with the same amount of accuracy. The present legal definitions owe their unsystematic character to the fact that they are ultimately based upon these rough equivalences. In the table appended I have given all the special units defined by or under the Act; a denomination within parentheses is not an Imperial denomination. The Act also allows a length to be specified in terms of any Imperial denomination and its decimal multiples and submultiples.

ART. 57.—**The F.P.S. System.** For scientific purposes the foot is the principal unit of length; and it is exclusively adopted in what is called the British system of absolute units. In that system the foot, pound, and second are used to define all the other dynamical units. Hence, for shortness, it is sometimes called the F.P.S. system.

ART. 58.—**Metric Standard of Length.** In the metric system of weights and measures, instituted in France in 1795, the standard unit of length is the distance between the ends of a rectangular bar of platinum called the *metre des Archives* preserved at Paris, the bar being at the temperature of melting ice. That standard distance is denominated the *metre.* Originally, the metre was defined as the ten-millionth part of a quadrant of the earth's meridian through Paris. To determine that length the arc from Dunkirk to Barcelona (9° 40′ 45″) was measured in terms of the then existing French standard of length, called the toise of Peru, and it was found to contain 551,584·72 toises. The meridian is

not an exact circle, but an ellipse with the polar axis for the shorter axis. From measurements previously made the ellipticity was taken to be $\frac{1}{334}$; and by computing from these data the length of the quadrant was found to be 5,130,740 toises. The platinum bar was constructed to equal the ten-millionth part of 5,130,740 toises, when at its standard temperature of melting ice. Another standard metre was deposited in the Observatory at Paris, and twelve iron copies were distributed to other countries.



Former French Standard, called the "Toise of Peru."

The *toise de Perou* was so called from its having been constructed to measure an arc of the meridian in Peru. The breadth, thickness, and ends of the measure are represented in the illustration. The toise is given by the distance on the bar between the points of the arrows *a* and *b*.

ART. 59.—**Relation of the Standard Metre to its Primary Definition.** According to Colonel Clarke's determinations of the size and figure of the earth, which are at present the most authoritative, the length of a quadrant of the meridian through Paris is 10,001,472 metres, and thus the standard metre is slightly less than the distance upon which it was founded. A difference of

that order does not impair the practical utility of the relation of the metre to the quadrant, but it shows that it is the material metre which is now to be considered as the ultimate standard.

According to the same authority the length of the polar axis is 500,482,296 inches; and Sir John Herschel pointed out that to define the inch as the five-hundred-millionth part of the polar axis would involve a correction which would be inappreciable in ordinary measurements, and sensible only with the nicest scientific instruments.

ART. 60.—**Derived Metric Units.** The great excellence of the metric system lies in the mode in which the subsidiary units are defined. Each is a decimal multiple or sub-multiple of the standard unit, and the scale of units is thus in harmony with the decimal nature of the notation of Arithmetic. The prefixes for multiples are derived from the Greek, and those for sub-multiples from the Latin.

Prefix.	Meaning as used.	Derivation.	Abbreviation.
Myria-	ten thousand,	μύριοι.	
Kilo-	one thousand,	χίλιοι.	k-
Hecto-	one hundred,	ἑκατόν.	h-
Deka-	ten,	δέκα.	da-
Deci-	one tenth,	decem.	d-
Centi-	one hundredth,	centum.	c-
Milli-	one thousandth,	mille.	m-

The abbreviations given are those authorized by the Comité International, which has for its object the development of the metric system. The abbreviation for metre is *m.*; hence *km.* for kilometre, *mm.* for millimetre.

ART. 61.—**Comparison of Yard and Metre.** The metric system has now been adopted over the greater part of Europe and has begun to spread over America. Its use in contracts was legalized in Britain in 1864, and it has also been legalized in the United States. A careful comparison of the yard and metre was made by Captain Kater, who gave as his result

$$1 \text{ metre} = 39{\cdot}37079 \text{ inches.}$$

This equivalence was adopted in the Act of 1864. A more recent determination by Colonel Clarke gives

$$1 \text{ metre} = 39{\cdot}37043 \text{ inches.}$$

Recently, Professor Stoney * has deduced a convenient equivalent

$$1 \text{ yard} = {\cdot}9144 \text{ metre},$$

which agrees with the above, so far as they do agree, and cuts off the uncertain difference.

These are determinations for the platinum metre at 32° Fahr., and the bronze yard at 62° Fahr. With a brass metre correct at 32° Fahr., but used at an ordinary temperature such as 62° Fahr., the equivalence is

$$1 \text{ metre} = 39{\cdot}382 \text{ inches.}$$

A sufficient approximation for ordinary purposes is

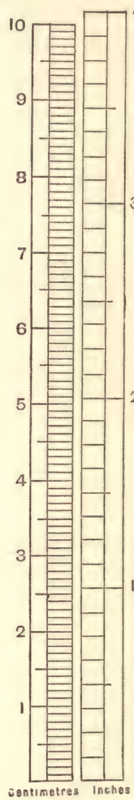
$$1 \text{ metre} = 39\tfrac{3}{8} \text{ inches;}$$

while a rougher approximation, easily remembered, is

$$1 \text{ decimetre} = 4 \text{ inches.}$$

In Fig. 1 the decimetre is compared with 4 inches.

Fig. 1.



ART. 62.—**The C.G.S. System.** The British Association, through their Committee on Electrical Standards, have extended the metric system and developed it so as to suit the measurement of electrical quantities. This development of the metric system is commonly called the Centimetre-Gramme-Second system, or by abbreviation

* *Nature*, vol. xxix., p. 278.

the C.G.S. system. The standard of length is the metre, but the primary unit of length is not the metre but the centimetre. This system was adopted as the basis for electrical measurements by the International Congress of Electricians, which met at Paris in 1881.

IMPERIAL MEASURES OF LENGTH.

1 *inch*		= 1/36 yard	= 25·4 millimetres.
1 *foot*	= 12 inches	= 1/3 yard	= 3·048 decimetres.
		1 YARD	= ·9144 metres.
1 *pole*, *rod*, or *perch*		= $5\frac{1}{2}$ yards	
1 *chain*	= 4 poles	= 22 yards	= 20·12 metres.
1 *furlong*		= 220 yards	
1 *mile*	= 8 furlongs	= 1760 yards	= 1·609 kilometres.

1 'fathom' = 2 yards; 100 'links' = 1 chain;
1 'hand' = 4 inches; 1 'ell' = 45 inches; 1 'pace' = 5 feet.

METRIC MEASURES OF LENGTH.

1 myriametre	= 10,000 metres.	
1 kilometre	= 1,000 metres.	
1 hectometre	= 100 metres.	
1 dekametre	= 10 metres.	
	1 METRE	= 39·371 inches.
1 decimetre	= ·1 metre.	
1 centimetre	= ·01 metre.	
1 millimetre	= ·001 metre.	
1 micron	= ·000001 metre.	

ART. 63.—**Change of Length.** When a straight rod is changed in length uniformly, the rate of expansion is expressed by

$$a \text{ L increment per L original length;} \qquad (1)$$

and the ratio of the expansion is expressed by

$$1 + a \text{ L expanded length per L original length.} \qquad (2)$$

The reciprocal of (1) is

$$\frac{1}{1+a}\,\mathsf{L} \text{ original length per } \mathsf{L} \text{ expanded length}; \qquad (3)$$

and the rate of diminution is

$$\left(1 - \frac{1}{1+a}\right) \mathsf{L} \text{ decrement per } \mathsf{L} \text{ expanded length}; \quad (4)$$

i.e.,
$$\frac{a}{1+a}\,\mathsf{L} \text{ decrement per } \mathsf{L} \text{ expanded length.}$$

When a is a small quantity, $1 - a$ is a sufficient approximation for $\frac{1}{1+a}$ (Art. 29), and then (3) becomes

$$(1 - a)\,\mathsf{L} \text{ contracted length per } \mathsf{L} \text{ original length};$$

and (4) becomes

$$a\,\mathsf{L} \text{ decrement per } \mathsf{L} \text{ original length.}$$

Here we have the same ideas as in the change in value of a sum of money discussed under simple interest.

EXAMPLES.

Ex. 1. Convert 123 kilometres into miles.

123 kilometres,

$$\begin{aligned} 1{,}000 \text{ metres} &= 1 \text{ kilometre}, \\ 39{\cdot}370 \text{ inches} &= 1 \text{ metre}, \\ 1 \text{ yard} &= 36 \text{ inches}, \\ 1 \text{ mile} &= 1{,}760 \text{ yards}; \end{aligned}$$

$$\therefore \quad \frac{123 \times 39370}{36 \times 1760} \text{ miles},$$

$$i.e., \quad \frac{41 \times 3937}{12 \times 176} \text{ miles},$$

i.e., 76·43 miles.

Ex. 2. Reduce 5 francs per metre to pence per yard, when exchange is 25 fr. 22 c. per £.

$$\begin{aligned} 240 \text{ pence} &= 1 \text{ pound}, \\ 1 \text{ pound} &= 25{\cdot}22 \text{ francs}, \end{aligned}$$

$$\begin{aligned} 5 \text{ francs} &= 1 \text{ metre,} \\ \cdot 9144 \text{ metres} &= 1 \text{ yard;} \\ \therefore \quad 240 \times 5 \times \cdot 9144 \text{ pence} &= 25 \cdot 22 \text{ yards,} \\ \textit{i.e.,} \quad \frac{109728}{2522} \text{ pence} &= 1 \text{ yard,} \\ \textit{i.e.,} \quad 43\tfrac{1}{2} \text{ pence} &= 1 \text{ yard.} \end{aligned}$$

EXERCISE IX.

1. Reduce 80 metres to yards, and 40 yards to metres.
2. Reduce 321 miles to kilometres.
3. Find the reciprocal of ·66 feet = link.
4. One metre contains 3·2809 English, and 3·1862 Prussian feet. By what fraction of an English inch does the Prussian exceed the English foot?
5. Determine the height of Mont Blanc in old Parisian feet, its height being given as 15,784 in English feet, and the ratio of the English to the old Parisian foot being as 1·0297 to ·9662.
6. Taking a centimetre at two fifths of an inch, find the number of centimetres in 30 yds. $3\frac{3}{8}$ inches.
7. Pliny says that the side of the base of the Great Pyramid was 883 *pedes*. If the *pes* is equivalent to the semi-cubit, what was the side of the Pyramid in English feet? Assume 1 cubit = 20·7 inch.
8. Three per cent. of the length of a wire is 4 inches too long for a certain purpose, and 3 feet 9 inches is ten per cent. too short. What is the length of the wire?
9. Reduce 4*s*. 6*d*. per yard to francs per metre when exchange is at 25 francs 22 c. per £.
10. Find the successive convergents to the ratio of the metre to the yard, taking 1 yard = ·9144 m.
11. The scale of a map is six inches to the mile; express the scale in terms of nch to inch.

SECTION X.—ANGLE.

ART. 64.—**Sexagesimal Units.** The circumference of the circle is divided into four equal parts called *quadrants*, and also into six equal parts called *sextants*. The sixtieth part of a sextant is denominated a *degree;* the sixtieth part of a degree is denomi-

nated a *minute;* and the sixtieth part of a minute is denominated a *second.* The sixtieth part of the second is denominated a *third,* but it is now the practice to divide the second decimally. This sexagesimal system of units has been in use ever since the time of the Egyptian astronomers.

An angle is not specified in terms of one of these units, but in terms of the series of units; thus $24° \ 35' \ 46''\cdot5$, where $° \ ' \ ''$ are abbreviations for degree, minute, second.

It is to be observed that an angle so specified is generally an *ordinal* quantity; the expression does not denote merely so much angle, but an angle of the specified amount measured from a fixed line of reference.

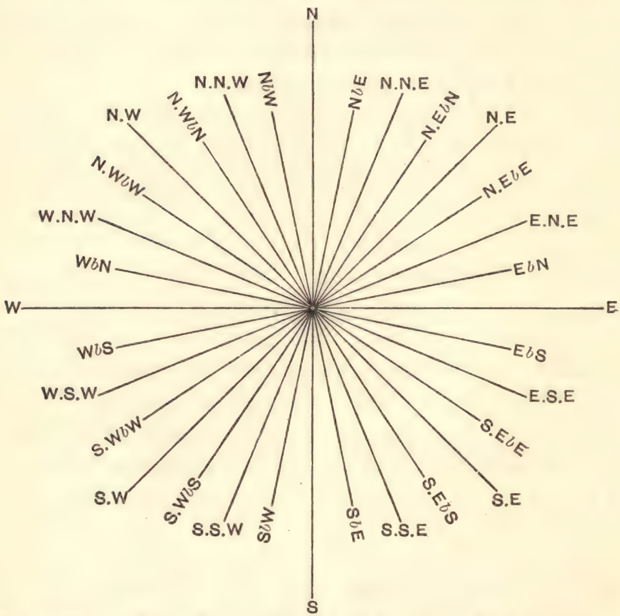
ART. 65.—**Centesimal Units.** The French reformers, when elaborating the metric system, introduced a centesimal division of the quadrant. They divided the quadrant into 100 equal parts, called *grades,* the grade into 100 *minutes,* and the minute into 100 *seconds.* This system, though intrinsically superior to the sexagesimal, has not been adopted even in France itself. Had the system proceeded upon a centesimal division of the degree instead of the quadrant it would have had a greater chance of being accepted.

ART. 66.—**Binary System.** The binary system is principally ordinal in its nature; that is, the terms are used to specify direction (Fig. 2). It is used in the mariner's compass.

The circle is first bisected by a north and south line; these directions are denoted by *N.* and *S.* The two semicircles are bisected into four quadrants, and the two new directions are denoted by *E.* and *W.* By bisecting each of the quadrants, four intermediate directions are obtained; each is denoted by means of the directions of the sides of the quadrant which it bisects, namely, *NE., SE., SW., NW.* By bisecting each of the octants eight more directions are obtained; each of these is also denoted

by the directions of the sides of the angle it bisects, namely, *NNE.*, *ENE.*, *ESE.*, *SSE.*, *SSW.*, *WSW.*, *WNW.*, *NNW.* By a further bisection, sixteen more directions are obtained; these, as shown in the diagram, are denoted on a slightly different plan.

Fig. 2.



The angle resulting from the five bisections is denominated a *point.* We evidently have 32 points = 1 revolution, 1 point = $11\frac{1}{4}$ degrees. The system is carried still further by subdividing the points into halves and quarters.

ART. 67.—**Circular Measure of an Angle.** The ratio connecting the circumference of a circle with its diameter is

$$\pi \text{ L arc per L diameter,}$$

where π denotes the incommensurable number 3·141593 –.* For exact calculations 3·1416 L arc = L diameter

* $\pi = 3{\cdot}14159265358979323846264338327 9+$.

is generally a sufficient approximation; while rougher convergents are

$$22 \text{ L arc} = 7 \text{ L diameter},$$
$$355 \text{ L arc} = 113 \text{ L diameter}.$$

These convergents may be derived from 3·1416 by the method of continued fractions explained in Art. 52. 51

The above is a ratio-rate, for the units on the two sides are the same. The value of the rate is in consequence independent of the size of the unit.

As the radius is half the diameter, we deduce

$$2\pi \text{ L arc} = \text{L radius}.$$

Any angle may be specified in terms of L arc per L radius, the value of one revolution being 2π. It has become customary to denote this ratio-unit L arc per L radius by the single word *radian*.

$$1 \text{ radian} = 57{\cdot}2958 \text{ degrees},$$
$$1 \text{ degree} = {\cdot}0174533 \text{ radians}.$$

Art. 68.—**Nautical Units of Length.** The sea-mile or knot is defined in terms of the circumference of the earth. As the earth is not an exact sphere, it is necessary to choose a particular circumference, or great circle, as it is called. The Admiralty knot is the length of a minute on the equatorial circumference. It is estimated to be equivalent to 6,086 feet, but its regulation equivalent is 6,080 feet. The common knot is the length of a minute on the mean meridian circumference; it is estimated to be equivalent to 6,076 feet.

1 league = 3 knots; 1 degree = 60 knots.

The following system of sub-units is coming into use:—

1 knot = 10 cables = 1,000 fathoms;

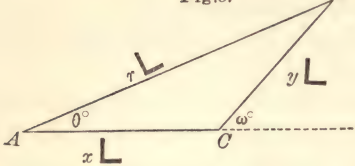
the fathom so defined differing slightly from the common fathom, which is equivalent to two yards.

The geographical mile is the same as the nautical mile.

Art. 69.—**Composition of a Vector in a Plane.** By a vector is meant a finite straight line having a specified direction. When our attention is restricted to a plane, it is necessary to specify one

angle only. Let AB be a vector of length r ∟, and direction $\theta°$. The two sides of any triangle which has AB for its third side are called the components of AB; as, for example, AC and CB. The direction of AB may be denoted by *along*, of AC by *adjacent*, and of CB by *opposite*.

Fig.3.



A vector is equivalent to the sum of its components, thus

r ∟ along $= x$ ∟ adjacent $+ \; y$ ∟ opposite. (1)

ART. 70.—**Sine and Cosine, Secant and Cosecant.** From the above complete equivalence certain partial equivalences can be derived, as in the case of Mixture (Art. 25). Thus

x/r ∟ adjacent $=$ ∟ along, (2)

and y/r ∟ opposite $=$ ∟ along. (3)

When the two components are at right angles to one another, the former rate (2) is called the *cosine*; and the latter (3) is called the *sine*. In that case the values are related by the equation

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$

When the components are inclined at an angle $\omega°$, the relation is

$$x^2 + y^2 + 2xy \cos \omega = r^2.$$

The reciprocals of (2) and (3) are

r/x ∟ along $=$ ∟ adjacent, (4)

and r/y ∟ along $=$ ∟ opposite. (5)

When the components are rectangular, the former is called the *secant*, and the latter the *cosecant*.

ART. 71.—**Tangent and Cotangent.** From (2) and (3) we derive an equivalence between the components, namely,

y/x ∟ opposite $=$ ∟ adjacent, (6)

with its reciprocal x/y ∟ adjacent $=$ ∟ opposite. (7)

When the components are rectangular, the former is called the *tangent*, and the latter the *cotangent*.

These are ratio-rates in the sense of their value being independent of the size of the unit of length; but the unit of length has a different direction in the two members of the rate.

ART. 72.—**Gradient.** By the gradient of a road is meant the rate connecting the amount of rise with the amount of advance. It is usually expressed as 1 in 100, 1 in 250, etc. This means

$$1 \text{ ft. } {}^{\text{fall,}}_{\text{rise,}} = 100 \text{ ft. advance,}$$
$$1 \text{ do.} = 250 \quad \text{do.}$$

It is the second number which is made the variable number.

The idea corresponds to that of *sine*, only the first direction is necessarily that of the vertical.

ART. 73.—**Composition of a Directed Quantity in Space.** When we consider tri-dimensional space, the components of a vector along mutually perpendicular directions are three in number. The partial equivalences of the first kind are called the direction-cosines of the vector. They may be, for example,

$$l \text{ L east} = \text{L along,}$$
$$m \text{ L north} = \text{L along,}$$
$$n \text{ L up} = \text{L along.}$$

The values l, m, n, are connected by the condition

$$l^2 + m^2 + n^2 = 1.$$

Each rate has its reciprocal form, and by combining any two we derive a rate of the tangent kind. Thus there are three rates of the tangent kind, with their three reciprocal forms.

ART. 74.—**Curvature and Radius of Curvature.** By the curvature of a curve lying in a plane is meant the rate at which the direction of the curve changes in passing along the curve. It is expressed in terms of

radian per L curve.

When the curve is a circle, the value of this rate is constant; hence the value may be deduced from the fact that in going once round the circle the direction changes by 360 degrees, that is, by

2π radians. If, then, the length of the circle is $c\,\mathsf{L}$, its curvature is

$$2\pi/c \text{ radian per } \mathsf{L} \text{ curve.}$$

But if we are given the radius of the circle as $r\,\mathsf{L}$, then its circumference is $2\pi r\,\mathsf{L}$, and the rate of curvature is expressed as

$$1/r \text{ radian per } \mathsf{L} \text{ curve.}$$

A portion of any curve, throughout which the curvature does not change sensibly, coincides with a portion of the arc of some circle. Hence, if the radius of that circle is $r\,\mathsf{L}$, the curvature for that portion of the curve will be

$$1/r \text{ radian per } \mathsf{L} \text{ curve.}$$

The radius of the equivalent circular arc is called the *radius of curvature.*

Examples.

Ex. 1. Express 12° 34′ 56″ in terms of radian; and 3 radians in terms of deg. min. sec.

$$\begin{array}{rrl}
1 \text{ degree} = & \cdot 0174533 & \text{radians.} \\
 & 12 & \\
\hline
 & \cdot 2094396 & \\
1 \text{ minute} = & 0\cdot 002909 & \text{radians.} \\
 & 34 & \\
\hline
 & 11636 & \\
 & 8727 & \\
\hline
 & \cdot 0098906 & \\
1 \text{ second} = & \cdot 0000048 & \text{radians.} \\
 & 56 & \\
\hline
 & 288 & \\
 & 240 & \\
\hline
 & \cdot 0002688 & \\
 & \cdot 0098906 & \\
 & \cdot 2094396 & \\
\hline
 & \cdot 2195990 &
\end{array}$$

Answer—·2196 radians.

Second part—

$$\begin{array}{rr} 1 \text{ radian} = & 57{\cdot}2958 \text{ degrees.} \\ & 3 \\ \hline & 171{\cdot}8874 \\ & 60 \\ \hline & 53{\cdot}2440 \\ & 60 \\ \hline & 14{\cdot}6400 \end{array}$$

Answer—170° 53′ 14″·28.

Ex. 2. Calculate from the primary definition of the metre, and the relation of the metre to the mile, the length of the diameter of the earth in miles.

40,000,000 metres circumf.,

113 metres diam. = 355 metres circumf.,

315 inches = 8 metres,

1 yard = 36 inches,

1 mile = 1760 yards;

$$\therefore \quad \frac{40{,}000{,}000 \times 113 \times 315}{355 \times 36 \times 1760 \times 8} \text{ miles diameter.}$$

$$\begin{array}{rr} 7{\cdot}60206 & 2{\cdot}55022 \\ 2{\cdot}05307 & 1{\cdot}55630 \\ 2{\cdot}49831 & 3{\cdot}24551 \\ \cline{1-1} & {\cdot}90309 \\ 12{\cdot}15344 & \\ \cline{2-2} 8{\cdot}25512 & 8{\cdot}25512 \\ \cline{1-1} 3{\cdot}89832 & \end{array}$$

Answer—7912 miles diameter.

Ex. 3. How much must a rail 6 ft. long be bent in order to fit into a curve of half a mile radius?

360 degrees = $\pi \times 1760$ yards arc,

2 yards arc;

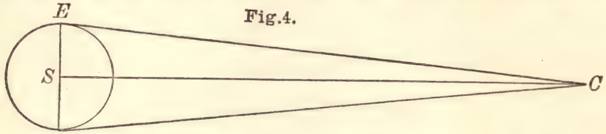
$$\therefore \quad \frac{360 \times 2}{\pi \times 1760} \text{ degrees,}$$

$$i.e., \quad \frac{9 \times 7}{22 \times 22} \text{ degrees,}$$

$$i.e., \quad \cdot 13 \quad ,,$$

Ex. 4. The radius of the earth's orbit subtends an angle of four-tenths of a second at 61 Cygni; how far is the star from the sun?

Fig.4.



Here the dependence of *ES* upon *SC* is given by the tangent of the angle *ECS*, that is, of $\cdot 4''$. For so small an angle the value of the radian is equal to the value of the tangent with sufficient accuracy for the present calculation.

$\cdot 01745$ mile arc per mile radius $= 1$ degree,

1 degree $= 3600$ seconds,

$\cdot 4$ seconds;

$$\therefore \quad \frac{\cdot 01745}{9000} \text{ mile arc} = \text{mile radius,}$$

$$9152 \times 10^4 \text{ miles arc;}$$

$$\therefore \quad \frac{9152 \times 10^4 \times 9 \times 10^3 \times 10^5}{1745} \text{ miles radius,}$$

$$i.e., \quad \frac{9152 \times 9}{1745} \times 10^{12} \text{ miles radius,}$$

$$i.e., \quad 472 \times 10^{11} \text{ miles radius,}$$

$$\text{or} \quad 4 \cdot 72 \times 10^{13} \text{ miles radius.}$$

Observation.—It is more common to put the point after the first figure, so that the index of 10 may be the mantissa of the logarithm of the number.

EXERCISE X.

1. Express $\frac{2}{15}$ of a right angle in degrees and also in grades.

2. The angle which is subtended by an arc equal to the radius is 206,264·8 seconds. Reduce this to grades and decimals, and thence deduce the value of $1/\pi$.

3. A circle has a circumference of 123 yards. What is the length of its radius?

4. How many radians are there in an arc of $114° 35' 30''$?

5. Express 45 degrees in terms of radian; and 4·5 radians in terms of degree.

6. Express *NW.* by *N.* and *SW.* by *W.* in degrees, reckoning east from north.

7. How many points and degrees are there between *N.* by *E.* and *SSE.*?

8. If the length of $\frac{1}{360}$ of the earth's circumference be $69\frac{1}{22}$ miles, find its diameter, taking diameter to circumference as 7 to 22.

9. From the primary definition of the metre and its relation to the statute mile deduce the average number of miles in a degree of latitude.

10. The cotangent of the angle of a roof is 1·5; what is the ratio of rise to span?

11. A railway line has a gradient of 1 in 80 for a distance of 3 miles; what is the total rise or fall?

12. The sines of two angles of a triangle are $\frac{3}{5}$ and $\frac{5}{13}$, and the side opposite to the former is 10 yards; required the side opposite the latter, and the third side.

13. The base of the Great Pyramid is a square 764 feet in the side, and the perpendicular height is 486 feet; required the length of the slant side of one of the triangular faces.

14. A man 5 feet 4 inches in height standing at a distance of 52 feet from the base of an electric-light tower casts a shadow 8 feet long on the pavement. What is the height of the tower?

15. The difference of longitude between two places is 5 degrees, and the latitude of both is 45 degrees; find the distance between them along the parallel of latitude. (Take the radius of the earth to be 4,000 miles.)

16. The length of a railway curve which has a uniform curvature is one mile, and the change of direction is 30 degrees. Find the value of the curvature and of the radius of curvature.

17. Express the sea-mile in terms of the kilometre.

18. Compare the nautical-fathom with the common fathom.

SECTION XI.—SURFACE.

ART. 75.—**General Unit of Surface.** The general unit of surface is appropriately denoted by **S**. It can be defined by means of the unit of length, and the mode commonly adopted is as follows: —The unit of surface is the area of the square which is formed with the unit of length as the side. When **S** is so defined, we have

$$1\,\mathbf{S} = \mathbf{L}^2,$$

where L^2 denotes square L. Such a unit of surface is called a systematic unit. When S is not so defined we have

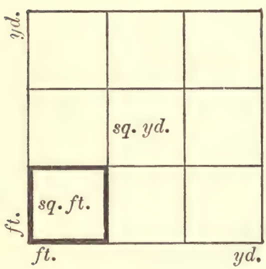
$$k\,\mathsf{S} = \mathsf{L}^2,$$

where k denotes some number, integral or fractional.

In what follows S is commonly restricted to being systematic.

ART. 76.—**Imperial Units of Surface.** In the imperial system the principal unit of surface is the area of the square formed by the yard, and denominated the *square yard.* Of the other units of surface some are derived systematically from the corresponding linear unit; and two, the *acre* and the *rood,* used in the measurement of land, are defined in terms of the square yard.

Fig. 5.



Since 3 feet long = 1 yard long,
and 3 feet broad = 1 yard broad,
it follows that

3 ft. long by 3 ft. broad = 1 yd. long by 1 yd. broad,

or 3^2 feet^2 = 1 yard^2;

i.e., 9 square feet = 1 square yard.

This is also evident from an inspection of the figure, where the longer side represents a yard and the shorter side a foot.

In science and the arts the square foot and the square inch are commonly used.

ART. 77.—**Metric Units of Surface.** In the metric system the unit of surface is derived from the area of the square formed by a unit of length. The natural unit is the area of the square formed by the metre, but it is an inconveniently small unit for the purpose of measuring land. Hence the special name of *are* was given, not to the area of the square formed by the metre, but to the area of the square formed by the dekametre, which is 100 times the former. From the *are* are derived, as will be seen from the table appended, the usual decimal multiples and submulti-

ples. This set of units is used in the measuring of land, and the *hectare* is the unit principally used.

For scientific purposes it is more convenient to use the square units given in the second column, in which any unit is equivalent to 100 of the next lower unit.

The authorized abbreviation for square is the index 2, as in *mm.*² for square millimeter. That for are is *a.*, giving *ha.* for hectare.

In the C.G.S. system the primary unit of surface is the square centimetre.

IMPERIAL MEASURES OF SURFACE.

1 sq. inch		= 1/1296 sq. yd.	
1 sq. foot		= 1/9 sq. yd.	
		1 SQUARE YARD	= 0·8361 sq. metres.
1 sq. pole or perch		= $30\frac{1}{4}$ sq. yds.	
1 sq. chain		= 484 sq. yds.	
1 *rood*	= 40 sq. poles	= 1,210 sq. yds.	
1 *acre*	= 4 roods	= 4,840 sq. yds.	= 0·4047 hectares.
1 sq. mile	= 640 acres	= 3,097,600 sq. yds.	= 259 hectares.

METRIC MEASURES OF SURFACE.

1 hectare	= 1 sq. hectometre	= 10,000 sq. metres.	
1 dekare		= 1,000 sq. metres.	
1 are	= 1 sq. dekametre	= 100 sq. metres.	
1 deciare		= 10 sq. metres.	
1 centiare	= 1 SQUARE METRE		= 1·196 sq. yds
1 milliare		= ·1 sq. metre.	
	1 sq. decimetre	= ·01 sq. metre.	
	1 sq. centimetre	= ·0001 sq. metre.	
	1 sq. millimetre	= ·000001 sq. metre.	

ART. 78.—**Area of a Rectangle.** When the unit of surface **S** is defined as the area of a square the side of which is **L**, the area of any rectangle is given by the rate

1 **S** per **L** long per **L** broad.

This rate may be written in the form of an equivalence in three ways—

$$\mathsf{1\,S} = \mathsf{L} \text{ long by } \mathsf{L} \text{ broad.}$$
$$\mathsf{1\,S} \text{ per } \mathsf{L} \text{ long} = \mathsf{L} \text{ broad.}$$
$$\mathsf{1\,S} \text{ per } \mathsf{L} \text{ broad} = \mathsf{L} \text{ long.}$$

For example, in the case of a web of cloth, the number of yards in the breadth is the same as the number of square yards per yard of length.

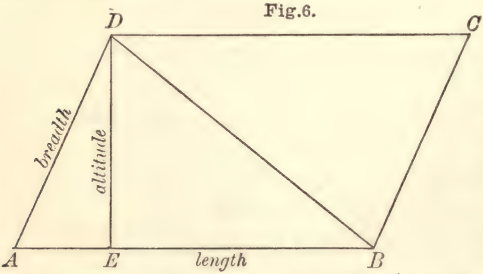
ART. 79.—**Signs for "by" and "per."** The sign $\times$ is in use to denote *by*. Its use in connecting units is in harmony with its use in connecting numerical quantities. According to recent scientific usage (Art. 11) the sign / is the appropriate sign for *per*, and that usage agrees with the expression % for *per cent.*

For example, take the particular case of the rate for a rectangle—

1 square yard / yard long / yard broad,
1 square yard = yard long $\times$ yard broad,
1 square yard / yard long = yard broad,
1 square yard / yard broad = yard long.

ART. 80.—**Area of a Parallelogram.** In a *parallelogram* the breadth is not in general perpendicular to the length; the component of the breadth which is perpendicular to the length is called the *altitude* (Fig 6). Hence

Fig.6.



$\mathsf{1\,S} \text{ per } \mathsf{L} \text{ long} = \text{alt.} \quad (1)$

If $\theta°$ denote the angle between the length and breadth, then

$\text{Sin}\,\theta\,\mathsf{L}\text{alt} = \mathsf{L}\text{brdth} \quad (2)$

If we eliminate the altitude between (1) and (2) we obtain

$\text{Sin}\,\theta\,\mathsf{S} = \mathsf{L}\text{lg. by } \mathsf{L}\text{bd.}$

ART. 81.—**Area of a Triangle.** The area of a *triangle* is half that of the corresponding parallelogram; the area of *ABD* is half that of *ABCD*. Hence for a right-angled triangle

$$\tfrac{1}{2}\mathsf{S} = \mathsf{L} \text{ long by } \mathsf{L} \text{ broad.}$$

This equivalence is of great importance in physical reasoning, as will be seen in subsequent chapters.

For a triangle of included angle $\theta°$

$$\frac{\sin\theta}{2}\mathsf{S} = \mathsf{L} \text{ long by } \mathsf{L} \text{ broad.}$$

EXAMPLES.

Ex. 1. Express 20£ per acre in terms of franc per hectare, taking the rate of exchange at 126 francs = 5£.

$$126 \text{ franc} = 5£,$$
$$20£ = \text{acre},$$
$$1000 \text{ acres} = 405 \text{ hectares};$$
$$\therefore \quad \frac{126 \times 4 \times 200}{81} \text{francs} = \text{hectare},$$
$$\textit{i.e.,} \quad \frac{11200}{9} \text{francs} = \text{hectare},$$
$$\textit{i.e.,} \quad 1244{\cdot}44 \text{ francs} = \text{hectare}.$$

Ex. 2. The height of the great pyramid was 486 feet. Assuming that its square was 8 Egyptian acres, the acre being the square of 100 royal cubits; required the length of the royal cubit.

$$486^2 \text{ foot}^2 = 8 \text{ Egyptian acres},$$
$$1 \text{ Egyptian acre} = 100^2 \text{ cubit}^2,$$
$$\therefore \quad \frac{486^2}{8 \times 100^2} \text{ foot}^2 = \text{cubit}^2;$$
$$\therefore \quad \frac{486}{2\sqrt{2} \times 100} \text{ foot} = \text{cubit},$$
$$\textit{i.e.,} \quad 1{\cdot}215 \times \sqrt{2} \text{ foot} = \text{cubit},$$

$$\begin{array}{r} 1{\cdot}4142 \\ 5121 \\ \hline 1414 \\ 283 \\ 14 \\ 7 \\ \hline 1{\cdot}718 \end{array}$$

Answer—1·72 feet = royal cubit.

Ex. 3. The area of a rectangle is 2,592 square feet. If the breadth were increased and the length diminished by 3 feet, we should have a square. Find the length and breadth.

For a rectangle

1 square foot = foot long by foot broad,

2592 square feet;

$\therefore$ 2592 foot long by foot broad,

x feet broad, $\therefore$ $\dfrac{2592}{x}$ feet long.

Now

$$x + 3 \text{ feet} = \frac{2592}{x} - 3 \text{ feet},$$

which reduces to the quadratic equation

$$x^2 + 6x - 2592 = 0;$$

$$\therefore \quad x = -3 \pm \sqrt{9 + 2592};$$

hence $x = 48$ is the possible solution.

Answer—48 feet broad, 54 feet long.

Ex. 4. A farm in the form of a square contains 400 acres; what is the length of a side expressed in terms of the mile?

1 mile^2 = 640 acres,

400 acres;

$\therefore$ $\frac{5}{8}$ mile^2,

i.e., $\frac{10}{16}$ mile^2,

i.e., $\dfrac{\sqrt{10}}{4}$ mile,

i.e., ·79 mile.

Ex. 5. Find the cost of papering the walls of a room 22 feet 6 inches long, 18 feet 9 inches wide, and 11 feet high; 99 square feet not requiring to be papered; the paper being 2 feet 9 inches wide, and costing 3*d.* per linear yard.

For a side

1 square foot = foot long by foot high,

22·5 ft. long by 11 ft. high;

$\therefore$ 22·5 × 11 square foot.

Hence for the two sides $2 \times 22 \cdot 5 \times 11$ square feet,
and for the two ends $2 \times 18 \cdot 75 \times 11$ „
Hence $2 \times 11\ (22 \cdot 5 + 18 \cdot 75) - 99$ square feet of paper,

$$1 \text{ foot long} = 2 \cdot 75 \text{ square feet,}$$
$$1 \text{ penny} = 1 \text{ foot long;}$$
$$\therefore \quad \frac{2 \times 11\ (22 \cdot 5 + 18 \cdot 75) - 99}{2 \cdot 75} \text{ pence,}$$

i.e., £1 4*s.* 6*d.*

EXERCISE XI.

1. The areas of England, Scotland, Ireland, are 58,311, 30,463, 31,754 square miles respectively ; express these areas in terms of the square kilometre.

2. A square, whose side is 146 feet, contains 1980·25 square metres. Find the number of inches in a metre to three decimal places.

3. Give that a metre equals 3·2809 ft.; find how many square metres there are in 1,000 square yards.

4. How many persons could be seated in an area ten miles square, each person occupying half a square metre?

5. What is 2*l.* per acre in terms of franc per hectare, when exchange is at 25 fr. 20 cts. per *l.*?

6. A plot of land is sold at 1,200*l.* per acre. What is the price in francs per square metre, taking 1*l.* = 25 francs?

7. Seventy-five per cent. of the area of a farm is arable ; of the remainder, eighty-five per cent. is pasture, and the rest is waste ; the area of the waste is 3 acres 0 roods 20 poles. What is the area of the farm?

8. Find the number of granite blocks required to pave a street which is 1 mile long and 16 yards wide, the block being 4 inches broad and 12 inches long.

9. Two rectangular fields are equal in area. The one is one third of a mile long by 990 feet wide ; the other is exactly square. Find the length of its side.

10. The sides of a rectangular piece of land are in the ratio of 2 to 7. What is the length of each side if the area contains 341,373,816 square feet?

11. The length of a rectangle is to its breadth as 4 to 3. If we increase the length by 3 feet and the breadth by 4, we increase the area by 287 feet. Required its dimensions.

12. The length of a rectangular piece of ground is twice its breadth ; its area is 30,479·805 square feet. Find its length and breadth.

13. Find the length of the side of a square which contains 1 acre.

14. Calculate to the nearest integer the number of feet in each side of a square field 4 statute acres in extent.

15. If 640 acres go to a square mile, what is the length of each side of a square piece of land which contains 100 acres?

16. Find the length of the side of a square field which contains 7 acres 3 roods 15 perches.

17. A room 21 feet long required 49 yards of carpet $\frac{3}{4}$ of a yard broad. Find the breadth of the room.

18. The number of yards of paper required to cover the four walls of a room 54 feet wide and 30 feet high is 880, and the breadth of the paper is $\frac{7}{8}$ of a yard. Required the length of the room.

19. The first of two pictures is 1 ft. 6 in. by 2 ft., the second 2 ft. by 2 ft. 6 in., and they are to be framed in the same way; if the glass and frame of the former cost 7*s.* 6*d.* and that of the latter 11*s.* 2*d.*, what is the price of the glass per square foot, and of the frame per foot of length?

20. A merchant buys cotton (27 inches in width) at 5 cents per square yard. He pays a duty of 2 cents per square yard, and 15 per cent. *ad valorem.* For what price per yard should he sell it in order to gain 25 per cent. on his outlay?

21. Of two squares of carpet, one measures 44 feet more round than the other, and 187 square feet in area. What are their sizes?

22. A square field of grain containing ten acres is to be cut down by a reaper working round and round; the cut of the reaper is 6 feet. How many rounds must the reaper take before the field is half cut?

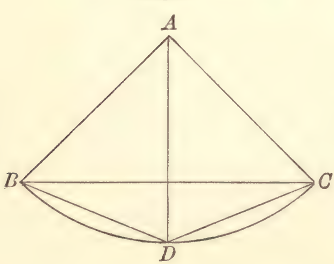
23. A square plot of ground, 21 yards in the side, is sold for the greatest number of sovereigns which can be placed flat upon it. Find the price, the diameter of the sovereign being seven eighths of an inch.

24. What length of fence is required to enclose 100 acres; *first*, when the land is in the form of a square; *second*, when in the form of a rectangle, having length $2\frac{1}{2}$ times breadth?

SECTION XII.—SURFACE, *Continued.*

ART. 82.—**Area of the Sector of a Circle.** Consider any sector of a circle ABC; draw the chord BC (Fig. 7). The area of the triangle ABC is a first approximation to the area of the sector ABC. Bisect the sector by the line AD; join BD, DC; the sum of the areas of the triangles ABD and ADC is a second approximation to the area of the sector.

Fig. 7.



Bisect each of these again; the sum of the areas of the four triangles will be a third approximation to the area of the sector. When the bisection has been continued a large number of times, the altitude of each of the triangles will not differ sensibly from the radius of the sector, and the sum of the lengths of the bases will not differ sensibly from the arc of the sector. Hence the area is given by the rate

$$\tfrac{1}{2}\,\mathsf{S} = \mathsf{L}\text{ arc by }\mathsf{L}\text{ radius.} \qquad (1)$$

If the angle of the sector is

$$\theta\,\mathsf{L}\text{ arc} = \mathsf{L}\text{ radius,} \qquad (2)$$

then, by multiplying these two equivalences together, L arc appears common to both sides, and may therefore be eliminated, and we obtain

$$\frac{\theta}{2}\,\mathsf{S} = (\mathsf{L}\text{ radius})^2.$$

ART. 83.—**Area of a Circle.** For a complete circle θ becomes 2π; hence for a circle

$$\pi\,\mathsf{S} = (\mathsf{L}\text{ radius})^2.$$

Since $\quad 2\,\mathsf{L}\text{ diameter} = \mathsf{L}\text{ radius,}$

$\therefore$ by squaring each side,

$$4\,(\mathsf{L}\text{ diameter})^2 = (\mathsf{L}\text{ radius})^2;$$

and

$$\frac{\pi}{4}\,\mathsf{S} = (\mathsf{L}\text{ diameter})^2.$$

A good approximation for $\frac{\pi}{4}$ is $\frac{11}{14}$.

ART. 84.—**Area of an Ellipse.** The circle described with the major axis AA' for diameter is called the auxiliary circle (Fig. 8). The area of that circle is given by

$$\frac{\pi}{4}\,\mathsf{S} = (\mathsf{L}\text{ major axis})^2.$$

The breadth of the ellipse is everywhere, as at PM, derived from the corresponding breadth of the circle, as QM, according to the ratio by which the minor axis BB' is derived from the diameter

of the circle DD'. Hence the area of the ellipse itself is given by

$$\frac{\pi}{4} \mathsf{S} = \mathsf{L} \text{ major axis by } \mathsf{L} \text{ minor axis,}$$

or $$\pi \mathsf{S} = \mathsf{L} \text{ semi-major axis by } \mathsf{L} \text{ semi-minor axis.}$$

Fig. 8.

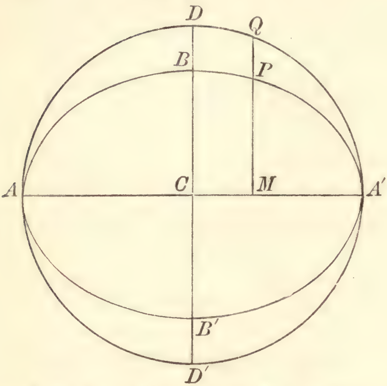
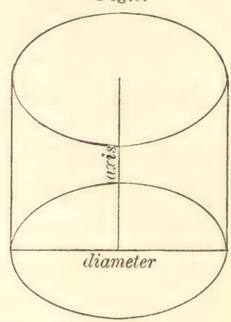


Fig. 9.



ART. 85.—**Surface of the Common Cylinder.** The flat portion of the surface consists of two equal and parallel circles (Fig. 9). If the curved surface were cut parallel to the axis, unrolled, and flattened out, it would form a rectangle having the circumference of the cylinder for length and the axis for breadth. Hence the area of the curved surface is given by

$$1 \mathsf{S} = \mathsf{L} \text{ circumference by } \mathsf{L} \text{ axis.}$$

The dependence of the circumference on the radius of the cylinder is given by

$$2\pi \mathsf{L} \text{ circumference} = \mathsf{L} \text{ radius};$$

$$\therefore \quad 2\pi \mathsf{S} \text{ by } \mathsf{L} \text{ circumf.} = \mathsf{L} \text{ circumf. by } \mathsf{L} \text{ axis by } \mathsf{L} \text{ radius,}$$

$$\textit{i.e.,} \quad 2\pi \mathsf{S} = \mathsf{L} \text{ axis by } \mathsf{L} \text{ radius.}$$

ART. 86.—**Surface of the Common Cone.** The flat portion of the surface is a circle (Fig. 10). If the curved surface were cut along the slant height, unrolled, and flattened out, it would form a sector of a circle, having the slant height for radius and the circumference of the base for arc. Hence (Art. 82) the area of its curved surface is given by

$\frac{1}{2}$ **S** = **L** circumference by **L** slant height;

or π **S** = **L** radius by **L** slant height.

Fig. 10.

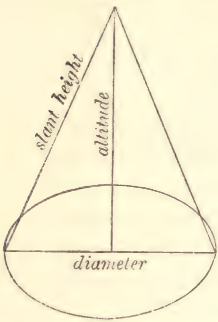
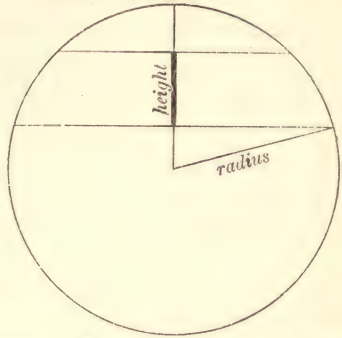


Fig. 11.



ART. 87.—**Surface of a Sphere.** The area of a spherical zone (Fig. 11) is given by

2π **S** = **L** radius of sphere by **L** height of zone.

Hence, as the height of a hemispherical zone is equal to the radius of the sphere, we have for a hemispherical surface

2π **S** = (**L** radius of sphere)^2.

Hence, for the area of a spherical surface,

4π **S** = (**L** radius)^2.

ART. 88.—**Solid Angle.** Just as a linear angle can be specified by means of the rate connecting the circumference of a circle with its radius (Art. 67), so a solid angle can be specified by means of the rate connecting the area of a surface of a sphere with the square of its radius. Thus a solid angle is specified in terms of

S per (**L** radius)^2;

and the unit-rate **S** per (**L** radius)^2 is sometimes called a *steradian*, that is, a solid radian.

To deduce the dependence of the area of a spherical lune on its spherical angle, 4π **S** per (**L** radius)^2 = 360 degrees;

$\therefore$ $\frac{\pi}{90}$ **S** per (**L** radius)^2 = degree.

ART. 89.—**Change of Surface.** Suppose that the length of a plate expands according to the rate

$$\alpha \text{ L increment per L original length,} \tag{1}$$

and its breadth according to the rate

$$\beta \text{ L increment per L original breadth,} \tag{2}$$

then the ratios of expansion are—

$$1+\alpha \text{ L expanded length} = \text{L original length,} \tag{3}$$

$$1+\beta \text{ L expanded breadth} = \text{L original breadth.} \tag{4}$$

Hence, if the included angle remain constant,

$$(1+\alpha)(1+\beta)\text{L}^2 \text{ expanded surface} = \text{L}^2 \text{ original surface,} \tag{5}$$

and

$$\{(1+\alpha)(1+\beta)-1\}\text{L}^2 \text{ increm. of surface} = \text{L}^2 \text{ orig. surface.} \tag{6}$$

The reciprocal of (5) is

$$\frac{1}{(1+\alpha)(1+\beta)}\text{L}^2 \text{ original surface} = \text{L}^2 \text{ expanded surface,} \tag{7}$$

and the rate of diminution,

$$\left(1-\frac{1}{(1+\alpha)(1+\beta)}\right)\text{L}^2 \text{ decrement} = \text{L}^2 \text{ expanded surface,} \tag{8}$$

When α and β are both small, the value of (5) is $1+\alpha+\beta$; of (6), $\alpha+\beta$; of (7), $1-\alpha-\beta$; and of (8), $\alpha+\beta$.

When β is equal to α, the value of (5) is $(1+\alpha)^2$, and its approximation is $1+2\alpha$; the value of (6) is $2\alpha+\alpha^2$, and of its approximation 2α; and so on.

EXAMPLES.

Ex. 1. The sides of a rectangle are 16 feet and 10 feet respectively. Find, to four places of decimals, the length of the diagonal of a square, whose area equals that of the rectangle.

16 feet long by 10 feet broad,

$$1 \text{ (foot side)}^2 = \text{foot long by foot broad,}$$

$$2 \text{ (foot diag.)}^2 = 1 \text{ (foot side)}^2;$$

$$\therefore \quad 16 \times 10 \times 2 \text{ (foot diag.)}^2,$$

$$\therefore \quad 8\sqrt{5} \text{ feet diag.};$$

i.e., $8 \times 2{\cdot}236068$ feet diag.,

i.e., $17{\cdot}888544$ „

Answer—17·8885 feet.

Observe—To get four places exact it is well to carry on the extraction of the root to six places.

Ex. 2. The diameter of a circle is 650 cm., what is its area?

$$11 \text{ sq. cm.} = 14 \text{ (cm. diam.)}^2,$$

650 cm. diam.

$$\therefore \quad \frac{11 \times (650)^2}{14} \text{ sq. cm.}$$

i.e., 331,964 „

i.e., 332×10^3 „

Ex. 3. How much will it cost to cover a circular plot of ground 130 feet in circumference with gravel at 4*d.* per square yard?

130 feet circumference,

$$1 \text{ foot rad.} = 2\pi \text{ feet circumference},$$

$$\pi \text{ sq. foot} = (\text{foot rad.})^2,$$

$$1 \text{ sq. yd.} = 9 \text{ sq. feet},$$

$$4d. = \text{sq. yd.}$$

Because we have (foot rad.)^2 the square of the preceding multipliers must be taken, hence

$$\frac{(130)^2 \times \pi \times 4}{(2\pi)^2 \times 9} \text{pence},$$

$$i.e., \quad \frac{(65)^2 \times 4}{9\pi} \quad ,,$$

Taking $\pi = \frac{22}{7}$, we find that the value is very nearly 600 pence, that is, £2 10*s.*

Ex. 4. Calculate the area of that zone of the earth's surface which lies between latitude 30° and latitude 45°, assuming the earth's radius to be 4,000 miles, and π to be 3·1416.

$$2\pi \text{ sq. miles} = \text{mile radius by mile altitude}.$$

For 45° we have

$$\frac{1}{\sqrt{2}} \text{ mile vertical} = \text{mile radius},$$

and for 30° $\frac{1}{2}$ mile vertical = mile radius;

hence $\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)$ mile altitude = mile radius;

4,000 miles radius;

$\therefore$ $4{,}000\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)$ mile altitude.

Hence

$$2 \times 3{\cdot}1416 \times 4{,}000 \times 4{,}000\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) \text{ sq. miles,}$$

i.e., $3{\cdot}1416 \times 16{,}000{,}000 \times (\sqrt{2} - 1)$ „

$$\begin{array}{r}
\cdot 41421 \\
61413 \\
\hline
124263 \\
4142 \\
1657 \\
41 \\
25 \\
\hline
1{\cdot}30128 \\
16 \\
\hline
780768 \\
130128 \\
\hline
2082048
\end{array}$$

Answer—$2{\cdot}082 \times 10^7$ square miles.

Ex. 5. With reference to a certain map it is known that a square inch represents $284\frac{4}{9}$ acres. What is the scale of the map?

$$2{,}560 \text{ acres} = 9 \text{ inch}^2,$$

$$1 \text{ mile}^2 = 640 \text{ acres};$$

$$\therefore \quad \frac{256}{9 \times 64} \text{ mile}^2 = \text{inch}^2;$$

$$\therefore \quad \sqrt{\frac{256}{9 \times 64}} \text{ mile} = \text{inch};$$

i.e., $\frac{16}{24}$ mile = inch,

i.e., 2 mile = 3 inch.

EXERCISE XII.

1. Find the area in acres of an isosceles right-angled triangle, the perpendicular from the right angle on the hypotenuse being 1,000 feet.

2. Find in hectares the area of a square of which the diagonal is 10 chains, assuming that a square metre is 1·196 of a square yard.

3. A rectangular enclosure is one acre in extent, and its perimeter is 322 yards. Find the lengths of its sides.

4. The diagonal of a square is 20 chains. Required the area in acres.

5. Find the area of a field which has the form of an equilateral triangle, and has one side 350 yards in length.

6. Calculate the area of the triangle whose sides are 11, 60, and 61 feet.

7. The two diagonals of a quadrilateral are 30 and 40 chains, and the angle between them is 30°. Required the area in acres.

8. A field is in the form of a trapezium, the two parallel sides being 6 chains and 4 chains; a third side is 5 chains, and is inclined to a parallel side at 60°. Calculate the area in acres.

9. A square bowling-green, 50 yards in the side, has a uniform slope round it, the slant height of which is 2 yards, and the angle which the slope makes with a horizontal plane is 30°. Find the cost of turfing the green and slope when the work is done at the rate of $\frac{1}{2}d.$ per square yard.

10. Find to the nearest square foot the number of square feet of lead required to cover a pyramidal roof, the base being a square of 19 feet in the side, and the height 6 feet.

11. The diameter of a circle is 135 mm. What is its circumference, and its area?

12. The circumference of a circle is 600·04 cm. What is its diameter and its area?

13. The area of a circle is 55,155 *ares*. What is its diameter and its circumference?

14. Express 36 circular inches in terms of square inches; and 63 square inches in terms of circular inches.

15. A penny and a halfpenny have diameters of one-tenth of a foot and of an inch respectively. If a halfpenny lie wholly on the top of a penny, what amount of the upper surface of the penny will be left uncovered?

16. An elliptic plot is described in a garden by means of a string 20 feet in length, and passing round two pegs distant by 5 feet. What is the area of the plot?

17. Calculate the area of an elliptical pond, the major axis of which is 250 feet, and the minor axis 150 feet.

18. Find the area of an ellipse whose axes are 25 chains and 22 chains 25 links.

19. Berlin is 11° 3·5′ east of Paris. Find the area of the portion of the earth's surface between their meridians; the earth's radius being taken as 3,963 miles.

20. On a certain map a square inch represents 4,000 acres. What is the linear scale?

21. The vertical scale of a drawing is 40 feet to an inch, and the horizontal 400 feet to an inch. What is the scale for the area?

22. On a map it is found that 100 acres are represented by 359·48 square inches; but the scale not being attached, it is required to calculate what it is. Give the scale in the form of a ratio, and also in terms of foot per mile.

SECTION XIII.—VOLUME.

ART. 90.—**General Unit of Volume.** Any unit of volume may be denoted by V. The systematic unit is the volume of a cube whose side is the unit of length, and in such case we have $1\ \mathsf{V} = \mathsf{L}^3$, where L^3 denotes the same as cubic L. When the unit is not systematic we have some other number instead of 1.

ART. 91.—**Imperial Units of Volume.** In the imperial system we have cubic units and units of capacity. The base of the cubic unit is the cubic yard, and the relations to it of the cubic foot and the cubic inch follow from the relations of the linear foot and the linear inch to the linear yard.

The primary unit of capacity is the *gallon*, which involves in its definition the standard of mass. It is the volume of ten imperial pounds of distilled water at the temperature of 62° Fahr. Further, the mass of the water is to be determined by weighing in air against brass weights, the air also being at the temperature of 62° Fahr., and the barometer standing at 30 inches.

In the original definition of the gallon, the volume defined as above was stated to be equivalent to 277·274 cubic inches; but when a more accurate determination of the density of water was made, the alternative part of the definition was repealed. According to the most recent determinations[1] the gallon is equivalent to 277·123 cubic inches. The brass gallon, marked "imperial

[1] Rankine's *Rules and Tables*, p. 99.

standard gallon," constructed when the gallon was originally defined, is not the ultimate standard, but pure water taken in conjunction with the standard of mass.

The other units of capacity are defined by means of the gallon.

ART. 92.—**Metric Units of Volume.** In the metric system we have three series of units of volume. The *stere* and its derivatives are for solid measure, as for example the measuring of wood; the *litre* and its derivatives are for fluid measure or measure of capacity; while the cubic series is the best adapted for calculations and for science generally. The *stere* is by definition equivalent to a cubic metre, and the *litre* to a cubic decimetre. Their derivatives are decimal; while those of the cubic series are millesimal.

The authorized abbreviation for cubic is the index^3, as in *cm.*^{3} for cubic centimetre. That for stere is *s.*, and for litre *l.*

In the C.G.S. system the primary unit of volume is the cubic centimetre.

IMPERIAL MEASURES OF VOLUME.

CUBIC.

	1 cubic inch	= 1/46656 cubic yard	
1 cubic foot	= 1,728 cubic inches	= 1/27 cubic yard	
		1 CUBIC YARD	= ·7645 cubic metre

CAPACITY.

1 *pint*		= 1/8 gallon	= ·568 litre
1 *quart*	= 2 pints	= 1/4 gallon	= 1·136 litres
		1 GALLON	= 4·544 litres
1 *peck*		= 2 gallons	
1 *bushel*	= 4 pecks	= 8 gallons	= ·3635 hectolitre
1 *quarter*	= 8 bushels	= 64 gallons	
1 *chaldron*	= 36 bushels	= 288 gallons	
	4 "gills"	= 1 pint	

FOR SALE OF DRUGS.

1 "minim"		= 1/480 "fluid ounce"	
1 "fluid drachm"	= 60 minims	= 1/8 fluid ounce	
		1 fluid ounce	= 1·8 cb. in. = ·0284 litre

METRIC MEASURES OF VOLUME.

SOLID.	LIQUID.	CUBIC.		
1 kilostere		= 1 cb. dekametre	= 1000 cubic metres	
1 hectostere			= 100 cubic metres	
1 dekastere			= 10 cubic metres	
1 stere	= 1 kilolitre	= 1 CUBIC METRE	= 1·308 cb. yd.	
1 decistere	= 1 hectolitre		= ·1 cubic metre	
1 centistere	= 1 dekalitre		= ·01 cubic metre	
	1 litre	= 1 cb. decimetre	= ·001 cubic metre	= ·22 gallons
	1 decilitre		= ·0001 cubic metre	
	1 centilitre		= ·00001 cubic metre	
		1 cb. centimetre	= ·000001 cubic metre	
		1 cb. millimetre	= ·000000001 cb. metre	

ART. 93.—**Volume of a Parallelepiped.** For a rectangular parallelepiped, V being defined as L^3,

$$1\ \mathsf{V} = \mathsf{L} \text{ long per } \mathsf{L} \text{ broad per } \mathsf{L} \text{ thick.}$$

When V is not defined as L^3, we have some number instead of 1.

This rate may be written as an equivalence,

$$1\ \mathsf{V} = \mathsf{L} \text{ long by } \mathsf{L} \text{ broad by } \mathsf{L} \text{ thick.}$$

When S is defined as L^2, we have (Art. 78)

$$1\ \mathsf{S} = \mathsf{L} \text{ long by } \mathsf{L} \text{ broad;}$$

$$\therefore\ 1\ \mathsf{V} = \mathsf{S} \text{ surface by } \mathsf{L} \text{ thick,}$$

or $\quad 1\ \mathsf{V}$ per S surface $= \mathsf{L}$ thick,

or $\quad 1\ \mathsf{V}$ per L long $= \mathsf{S}$ cross-section.

For example, in a rectangular slab the number of feet in the thickness is the same as the number of cubic feet per square foot of surface. In the case of a rectangular rod, the number of square inches in the cross section is the same as the number of cubic inches per inch of length.

When the parallelepiped is not rectangular, let the length and breadth be inclined at an angle of $\theta°$, and the thickness be inclined to the base at an angle of $\phi°$, then

$$\sin\theta \sin\phi\ \mathsf{V} = \mathsf{L} \text{ long by } \mathsf{L} \text{ broad by } \mathsf{L} \text{ thick.}$$

EXAMPLES.

Ex. 1. Given the rate of exchange 25·22 francs per £, and that 4·54 litres are equivalent to 1 gallon; deduce the relation of francs per litre to £ per gallon.

$$25{\cdot}22 \text{ francs} = 1£$$
$$4{\cdot}54 \text{ litres} = 1 \text{ gallon};$$

∴ dividing the one equivalence by the other,

$$\frac{2522}{454} \text{ francs per litre} = 1£ \text{ per gallon},$$

i.e., $$5{\cdot}55 \text{ francs per litre} = £ \text{ per gallon}.$$

Ex. 2. A schoolroom is 40 ft. long, 26 ft. 6 in. broad, and 19 ft. 3 in. high. If 80 cubic feet of space and 8 square feet of floor must be provided for each scholar, what is the maximum number of scholars which the room can provide for ?

40 ft. by 26·5 ft. by 19·25 ft.,

$$1 \text{ cubic ft.} = \text{ft. by ft. by ft.},$$
$$1 \text{ scholar} = 80 \text{ cubic feet};$$
$$\therefore \quad \frac{40 \times 26{\cdot}5 \times 19{\cdot}25}{80} \text{ scholars},$$

i.e., $$255 \text{ scholars}.$$

Again,

40 ft. long by 26·5 ft. broad,

$$1 \text{ sq. ft.} = \text{ft. long by ft. broad},$$
$$1 \text{ scholar} = 8 \text{ sq. ft.};$$
$$\therefore \quad 5 \times 26{\cdot}5 \text{ scholars},$$

i.e., $$132 \text{ scholars}.$$

Hence the greatest number of scholars provided for is 132.

EXERCISE XIII.

1. Express the litre in terms of the cubic foot.
2. An imperial gallon is 277·274 cubic inches ; a Winchester bushel 2150·42 cubic inches ; how many Winchester bushels are equal to 100 imperial bushels ?
3. Reduce 85 litres to gallons, and 54 gallons to litres.
4. By how much does a quart exceed a litre ?

5. Reduce 234 hectolitres to bushels, and 432 bushels to hectolitres.

6. What is 1·05 dollars per bushel in terms of shilling per quarter?

7. What is one shilling and sixpence a gallon in francs per litre, taking exchange at 25 francs per £?

8. A gas stove burns 7 cubic feet per hour, and the cost of the gas is 4*s.* per 1,000 cubic feet. Find the hourly cost.

9. A rectangular block of stone is as broad as it is long, and contains a cubic feet. If it were as broad as it is high the bulk would be b cubic feet. Find the length.

10. If a wall, 42 feet long, 10 feet high, and $2\frac{1}{2}$ feet thick, contains 12,800 bricks, how many bricks of the same kind will be required for a wall 112 feet long, 6 feet high, and 2 feet thick?

11. If 81 gallons of water will fill a cistern 4 feet 4 inches long, 2 feet 8 inches broad, and 1 foot $1\frac{1}{2}$ inch deep, how many cubic inches are contained in a pint?

12. The price of wheat is 36*s.* 6*d.* per quarter; express it in terms of francs per hectolitre, the rate of exchange being 25·30 francs = £.

13. The price of brandy is 52 centimes per litre; express it in pence per gallon, the rate of exchange being 25·80 francs = £.

14. By taking the decimetre as equal to four inches, what percentage of error is introduced, *first*, in linear measure; *second*, in square measure; *third*, in cubic measure. Take the legal equivalent as the true length of the metre?

15. Find the length of the side of a cubical vessel that shall contain twice as much as one whose side is 8 inches.

SECTION XIV.—VOLUME, *Continued.*

ART. 94.—**Volume of a Cylinder, a Cone, a Pyramid.** The volume of a cylinder is given by

$$1\ \mathsf{V} = \mathsf{S}\text{ base by }\mathsf{L}\text{ altitude,}$$

where V denotes L^3 and S denotes L^2.

In the case of a cone the altitude is variable, but it can be shown that the average altitude is $\frac{1}{3}$ of the greatest altitude. Hence

$$\tfrac{1}{3}\ \mathsf{V} = \mathsf{S}\text{ base by }\mathsf{L}\text{ greatest altitude.}$$

In a pyramid the altitudes vary in the same manner as in a cone. Hence

$$\tfrac{1}{3}\ \mathsf{V} = \mathsf{S}\text{ base by }\mathsf{L}\text{ greatest altitude.}$$

ART. 95.—**Volume of a Sphere.** A sphere may be considered as made up of a great number of pyramids having their bases on

the surface of the sphere, and their apices at its centre. Hence, from $\frac{1}{3}\,\mathsf{V} = \mathsf{S}$ base by L altitude,
and $4\pi\,\mathsf{S}$ surface $= (\mathsf{L}$ radius$)^2$,
since the surface and the bases, and the radius and the altitude coincide ultimately, we deduce

$$\frac{4\pi}{3}\,\mathsf{V} = (\mathsf{L}\text{ radius})^3.$$

Hence

$$\frac{\pi}{6}\,\mathsf{V} = (\mathsf{L}\text{ diameter})^3,$$

for which, approximations are

$$11\,\mathsf{V} = 21\,(\mathsf{L}\text{ diameter})^3,$$

and

$$377\,\mathsf{V} = 720\,(\mathsf{L}\text{ diameter})^3,$$

and

$$4{\cdot}1888\,\mathsf{V} = (\mathsf{L}\text{ diameter})^3.$$

ART. 96.—**Volume of an Ellipsoid.** The volume of an ellipsoid is given by

$$\frac{\pi}{6}\,\mathsf{V} = \mathsf{L}\text{ long diameter by }\mathsf{L}\text{ broad diameter by }\mathsf{L}\text{ thick diameter.}$$

In the case of a spheroid, two diameters are equal. If the two equal diameters are each greater than the third, the spheroid is said to be *oblate*, and

$$\frac{\pi}{6}\,\mathsf{V} = (\mathsf{L}\text{ long diameter})^2\text{ by }\mathsf{L}\text{ thick diameter.}$$

If the two equal diameters are less than the third, the spheroid is said to be *prolate*, and

$$\frac{\pi}{6}\,\mathsf{V} = \mathsf{L}\text{ long diameter by }(\mathsf{L}\text{ broad diameter})^2.$$

ART. 97.—**Change of Volume.** A volume may expand in three independent directions. Suppose that the several ratios of linear expansion are

$$1+\alpha\ \mathsf{L}\text{ expanded length} = \mathsf{L}\text{ original length,}$$
$$1+\beta\ \mathsf{L}\text{ expanded breadth} = \mathsf{L}\text{ original breadth,}$$
$$1+\gamma\ \mathsf{L}\text{ expanded depth} = \mathsf{L}\text{ original depth;}$$

then, if the included angles remain constant,

$$(1+\alpha)(1+\beta)(1+\gamma)\mathsf{L}^3 \text{ expanded vol.} = \mathsf{L}^3 \text{ original vol.,} \qquad (1)$$

and

$$\{(1+\alpha)(1+\beta)(1+\gamma)-1\}\mathsf{L}^3 \text{increment} = \mathsf{L}^3 \text{ original vol.} \qquad (2)$$

The reciprocal of (1) is

$$\frac{1}{(1+\alpha)(1+\beta)(1+\gamma)}\mathsf{L}^3 \text{ original vol.} = \mathsf{L}^3 \text{ expanded vol.}; \quad (3)$$

and the rate of diminution

$$1-\frac{1}{(+\alpha)(1+\beta)(1+\gamma)}\mathsf{L}^3 \text{ decrement} = \mathsf{L}^3 \text{ expanded vol.} \quad (4)$$

If α, β, γ is each a small fraction, the value of (1) may be taken as $1+\alpha+\beta+\gamma$; of (2) as $\alpha+\beta+\gamma$; of (3) as $1-\alpha-\beta-\gamma$; and of (4) as $\alpha+\beta+\gamma$.

When, further, $a=\beta=\gamma$, the values of (2) and (4) become $3a$, that is, 3 times the value of the linear rate of expansion.

EXAMPLES.

Ex. 1. Calculate the volume of a granite monument, consisting of a right cylindrical shaft 8 feet high, surmounted by a right circular cone 5 feet high, the common radius of the cone and cylinder being $2\frac{1}{2}$ feet. (Take $\pi = \frac{355}{113}$.)

For a circular cylinder,

$$\pi \text{ cubic feet} = (\text{ft. rad.})^2 \text{ by ft. high,}$$

$(2{\cdot}5)^2$ (ft. rad.)^2 by 8 ft. high;

$$\therefore \qquad \pi \times (2{\cdot}5)^2 \times 8 \text{ cubic feet.}$$

For a circular cone,

$$\frac{\pi}{3} \text{ cubic feet} = (\text{ft. rad.})^2 \text{ by ft. high,}$$

$(2{\cdot}5)^2$(ft. rad.)^2 by 5 ft. high;

$$\therefore \qquad \frac{\pi}{3} \times (2{\cdot}5)^2 \times 5 \text{ cubic feet.}$$

Hence the total volume is

$$\pi(2{\cdot}5)^2(8+\tfrac{5}{3}) \text{ cubic feet,}$$

$$i.e., \quad \frac{355 \times (2{\cdot}5)^2 \times 29}{113 \times 3} \text{ cubic feet.}$$

Take logs of the upper factors by themselves and of the lower factors by themselves, add each column, and subtract.

$$\begin{array}{r} 2{\cdot}55023 \\ 0{\cdot}39794 \\ 0{\cdot}39794 \\ 1{\cdot}46240 \\ \hline 4{\cdot}80851 \\ 2{\cdot}53020 \\ \hline 2{\cdot}27831 \end{array} \qquad \begin{array}{r} 2{\cdot}05308 \\ 0{\cdot}47712 \\ \hline 2{\cdot}53020 \end{array}$$

Answer—189·8 cubic feet.

Ex. 2. A circular plate of lead, 2 inches in thickness and 8 inches in diameter, is converted without loss into spherical shot of the same density, and each of ·05 inch radius. How many shot does it make ?

As the density of the plate and of the shot is the same, we require to consider the volume of the materials only.

For the plate,

$$\pi \text{ cubic inch} = (\text{inch rad.})^2 \text{ by inch thick,}$$

16 (inch rad.)^2 by 2 inch thick ;

$$\therefore \quad 32\pi \text{ cubic inches.}$$

For the shot,

$$\tfrac{4}{3}\pi \text{ cubic inch} = (\text{inch rad.})^3,$$

$$\cdot 05 \text{ inch rad. ;}$$

$$\therefore \quad \tfrac{4}{3}\pi(\tfrac{1}{20})^3 \text{ cubic inch} = 1 \text{ shot,}$$

$$32\pi \text{ cubic inch ;}$$

$$\therefore \quad \frac{32 \times (20)^3 \times 3}{4} \text{ shot,}$$

i.e, 192,000 shot.

EXERCISE XIV.

1. Find the volume of a cone whose altitude is 2 feet, and the diameter of the base 1 foot 6 inches.

2. Find the surface of a sphere which is one yard in diameter.

3. Find the radius of the sphere the volume of which is equal to the sum of the volumes of two spheres, whose radii are 3 feet and 4 feet.

4. The area of the base of a cylinder is 2 square feet and its height 30 inches; find the height of a cylinder the solid content of which is three times as great, but whose diameter is only two thirds of the given one.

5. If the volume of the first of two cylinders is to that of the second as 10 to 8, and the height of the first is to that of the second as 3 to 4, and if the base of the first has an area of 16·5 square feet, what is the area of the base of the second?

6. Determine the volume of the earth, supposing its diameter to be 8,000 miles. How many masses of the size of the earth would make up one of the size of the sun, the diameter of which is 880,000 miles?

7. Two spheres, *A* and *B*, have for radii 9 feet and 40 feet; the superficial area of a third sphere *C* is equal to the sum of the areas of *A* and *B*. Calculate the excess in cubic feet of the volume of *C* over the sum of the volumes of *A* and *B*.

8. A cone and hemisphere being supposed to have a common base and to lie at opposite sides of it; required, the ratio of the altitude of the cone to the radius of the hemisphere, in order that the volumes of the two solids should be equal.

9. Determine approximately the length of the radius of the sphere whose volume is 400 cubic feet.

10. A sphere and a cube have an equal amount of surface; what is the ratio of their volumes?

11. If a model is made on the linear scale of 1/40 inch to the foot; what is the scale for surface and for volume?

12. The altitude of a common cone equals the circumference of its base. Calculate the volume and the area of the whole surface of the cone, the radius of the base being 6 inches.

13. The interior of a building is in the form of a cylinder of 30 feet radius and 12 feet altitude, surmounted by a cone whose vertical angle is a right angle. How many cubic feet of air will it contain?

14. The long axis of a spheroid is 10 inches, and each short axis 6 inches. Find its volume.

CHAPTER THIRD.

KINEMATICAL.

SECTION XV.—TIME.

Art. 98.—**Sidereal Units.** The idea of time is fundamental. The general unit is appropriately denoted by T.

The standard of time adopted by all civilized nations is the time occupied by the earth in making one rotation about its axis. This interval is marked out by the successive transits of a particular star across the meridian of a place, and it is on that account denominated a *sidereal day*. The sidereal day is divided into 24 *sidereal hours*, the sidereal hour into 60 *sidereal minutes*, and the sidereal minute into 60 *sidereal seconds*. The sidereal units are used principally by astronomers.

Art. 99.—**Mean Solar Units.** By a *year* is meant the constant interval occupied by the earth in making a revolution round the sun; it is marked out by the sun leaving and returning to a certain position among the stars. The transit of the centre of the sun across the meridian of a place marks out an interval called the *apparent solar day*. This apparent day is not completely constant; its length goes through a cycle of small changes in the course of a year. Its average length for the course of a year is called the *mean solar day*. This interval of the mean solar day is measured out by clocks and chronometers, corrected by observation, on the part of astronomers, of sidereal time.

The mean solar day is divided into 24 *mean solar hours*, each of which is divided into 60 *mean solar minutes*, each of which is divided into 60 *mean solar seconds*. When the terms "hour," "minute," "second" are used without qualification, "mean solar" is understood. In science the "second" is the unit principally used; it is chosen for the unit of time both in the British system of units and in the C.G.S. system of units.

$$1 \text{ mean solar day} = 1{\cdot}00274 \text{ sidereal day.}$$

$$1 \text{ mean solar second} = 1{\cdot}00274 \text{ sidereal second.}$$

ART. 100.—**Relation of Mean Solar Day to Year.** The interval occupied by the earth in making one rotation has not a simple relation to the interval occupied by the earth in making one circuit round the sun. Hence the equivalence of the year to the mean solar day involves a complex number. It is

$$1 \text{ year} = 365{\cdot}2422 \text{ mean solar days.}$$

The first approximation in use is

$$1 \text{ year} = 365 \text{ days};$$

it is called the common year, and is sufficient for all ordinary calculations. But it is not sufficient for the purpose of arranging the calendar, so that a given date of the month shall always fall on the same season of the year. For this purpose a second approximation was introduced under the authority of Julius Cæsar:

$$4 \text{ years} = 4 \times 365 + 1 \text{ days,}$$

i.e., $$1 \text{ year} = 365{\cdot}25 \text{ days.}$$

This equivalent is called the Julian year.

A third approximation is

$$25 \times 4 \text{ years} = 25(4 \times 365 + 1) - 1 \text{ days,}$$

or $$1 \text{ year} = 365{\cdot}24 \text{ days.}$$

A fourth approximation is

$$4 \times 25 \times 4 \text{ years} = 4\{25(4 \times 365 + 1) - 1\} + 1 \text{ days,}$$

or $$1 \text{ year} = 365{\cdot}2425 \text{ days.}$$

This equivalent is called the Gregorian year, because the approximation was applied to the calendar under the authority of Pope Gregory.

The mode in which the Gregorian equivalence is applied is as follows :—

A year whose number is not divisible by 4 contains 365 days.

A year whose number is divisible by 4, but not by 25, contains 366 days.

A year whose number is divisible by 4 and by 25, but not by 4 again, contains 365 days.

A year whose number is divisible by 4 and by 25, and by 4 again, contains 366 days.

ART. 101.—**Epoch and Era.** When we come to specify time as an ordinal quantity we require to choose an origin from which to reckon. The *civil* day is reckoned from mean midnight to mean midnight; the *nautical* and the *astronomical* from mean noon to mean noon. The two latter differ in this respect, that the number for a civil day is by the nautical reckoning given to the interval between the preceding noon and the noon of the civil day; whereas by the astronomical reckoning it is given to the interval between the noon of the civil day and the succeeding noon. Astronomers reckon the hours up to 24, and this mode of reckoning is sometimes found more convenient on extensive railway systems.

By the epoch of an era is meant the particular year from which the years are numbered, that of the Christian era being the year of the birth of Christ.

ART. 102.—**Local Time and Standard Time.** The civil day is reckoned from mean noon, but each meridian has its own mean noon; hence the same instant is not denoted by the same number at places on widely different meridians. The connection between difference of time and difference of longitude is given by

24 hours later = 360 degrees of longitude west,

or 1 hour later = 15 degrees of longitude west.

In the British Islands the legal origin for the civil day at a place is the mean midnight at the place; but it is found more

convenient to use one standard of time throughout Great Britain, and the standard naturally adopted is the time for the meridian of Greenwich.

A ship reckons time from its own varying mean noon.

ART. 103.—**American Standard Time.** Recently the problem of how best to arrange the origin of the civil day was brought into prominence in North America owing to the construction of the Pacific Railways, which cross many degrees of longitude. The problem has been solved as follows:—The whole of North America has been divided into five broad belts running north and south, each extending over 15 degrees of longitude. In each belt one standard time will be maintained, the difference of one hour existing between two contiguous belts.

COMPARATIVE TIME.

Later than Greenwich.				Earlier than Greenwich.			
	H.	M.	S.		H.	M.	S.
Oxford, - - - - -	0	5	2	Cambridge, - - - -	0	0	23
Liverpool, - - - - -	0	12	17	Paris, - - - - - -	0	9	21
Edinburgh, - - - -	0	12	44	Brussels, - - - - -	0	17	29
Madrid, - - - - -	0	14	45	Rome, - - - - - -	0	49	55
Glasgow, - - - - -	0	17	11	Berlin, - - - - - -	0	53	35
Dublin, - - - - -	0	25	22	Vienna, - - - - -	1	5	31
Lisbon, - - - - -	0	36	35	Cape Town, - - - -	1	13	55
Madeira, - - - - -	1	7	36	Constantinople, - - -	1	55	57
Rio de Janeiro, - - -	2	52	41	St. Petersburg, - - -	2	1	13
Buenos Ayres, - - -	3	53	84	Cairo, - - - - - -	2	5	2
Halifax, N.S., - - -	4	14	24	Jerusalem, - - - -	2	20	59
Santiago de Chili, - -	4	42	42	Moscow, - - - - -	2	30	17
Quebec, - - - - -	4	44	49	Madras, - - - - -	5	20	29
New York, - - - -	4	55	57	Calcutta, - - - - -	5	54	0
Washington, - - - -	5	8	12	Pekin, - - - - - -	7	45	52
Chicago, - - - - -	5	50	27	Tokio, - - - - - -	9	18	40
San Francisco, - - -	8	9	45	Sydney, - - - - -	10	4	47
Fiji Islands, - - - -	11	52	0	Wellington, N.Z., - -	11	37	16

Art. 104.—Rate of Working; Resistance, Facility. In the solution of problems on the time required to accomplish a piece of work, several assumptions are made. It is assumed that for each workman or kind of workman there is a definite rate of working which is independent of the number of hands engaged, and is also uniform, though the length of the working day varies. Without constant or approximately constant rates of working to reason from, nothing can be concluded.

Suppose that the piece of work is reaping a field of grain, each hand working independently. Suppose that a man could do it in p hours, working uniformly, though not necessarily continuously. Then the *resistance* which the reaping of the field offers to a man is

$$p \text{ hours} = \text{field of grain.}$$

Here we use "field of grain" as an expression for the temporary unit "reaping the field of grain in question."

His *facility* of working, which is the reciprocal of the *resistance* of the field, is

$$\frac{1}{p} \text{ field of grain} = \text{hour.}$$

Art. 105.—Collective Facility, Resulting Resistance. Let the resistance of the field of grain to a woman, boy, girl be respectively q, r, s hours = field of grain. Suppose a men, b women, c boys, d girls set to work simultaneously, then the *collective facility* will be

$$\frac{a}{p}+\frac{b}{q}+\frac{c}{r}+\frac{d}{s} \text{ field of grain} = \text{hour};$$

and the *resulting resistance*, which is the reciprocal, will be

$$1\Big/\left(\frac{a}{p}+\frac{b}{q}+\frac{c}{r}+\frac{d}{s}\right) \text{ hours} = \text{field of grain.}$$

When water flows at constant rates into or out of a cistern, we have the same kind of ideas; only if a current inwards is reckoned a positive facility towards filling the cistern, a simultaneous current outwards is a negative facility.

Similar ideas are encountered when we consider the flow of electricity.

Examples.

Ex. 1. Washington Time is 5h. 8m. 12s. later than Greenwich Time; what is the longitude of Washington relatively to Greenwich?

$$15 \text{ unit of angle} = \text{corresponding unit of time},$$
$$5\text{h. } 8\text{m. } 12\text{s.}, \quad \therefore \quad 15\,(5^\circ\ 8'\ 12'');$$
$$i.e., \quad 77^\circ\ 3'\ \text{W}.$$

Ex. 2. A besieged garrison has sufficient provisions to last it for 23 weeks at the rate of 18 oz. per man per diem, but receiving a reinforcement of 40 per cent. upon its original number, the allowance is reduced to 15 oz. per man per diem; how many days will it be able to hold out?

$$18 \text{ oz. per man per day},$$
$$23 \times 7 \text{ day},$$
$$\therefore \quad 23 \times 7 \times 18 \text{ oz. per man};$$
$$n \text{ men},$$
$$\therefore \quad 23 \times 7 \times 18n \text{ oz.} \qquad (1)$$
$$15 \text{ oz. per man per day},$$
$$n(1 + \tfrac{2}{5}) \text{ men},$$
$$\therefore \quad n(1 + \tfrac{2}{5})\, 15 \text{ oz. per day},$$
$$\therefore \quad \text{from (1)} \quad \frac{23 \times 7 \times 18n}{n(1 + \frac{2}{5})15} \text{ days},$$
$$i.e., \ 138 \text{ days}.$$

Ex. 3. A can perform a piece of work in 12 days, B in 15 days, and C in 18 days, when working separately. Find the time in which they could perform it when working together.

Rate of working of	A,	$\frac{1}{12}$	piece of work	= day,
,,	B,	$\frac{1}{15}$	,,	= ,,
,,	C,	$\frac{1}{18}$	,,	= ,,
,,	A, B, and C,	$\frac{1}{12} + \frac{1}{15} + \frac{1}{18}$	,,	= ,,
				1 piece of work;

$$\therefore \quad \frac{1}{\frac{1}{12}+\frac{1}{15}+\frac{1}{18}} \text{ day,}$$

$$\textit{i.e.,} \quad \tfrac{180}{37} \text{ day,}$$

$$\textit{i.e.,} \quad 4{\cdot}86 \text{ working days.}$$

Ex. 4. Suppose that every hour per day that a student works requires 30 days of rest during the year; how many hours per day must he read so as to do the greatest amount possible?

Suppose hours of work = day of work; (1)

$x30$ days of rest = year,

$\therefore$ $365 - x30$ days of work = year. (2)

Therefore from (1) and (2),

$x(365 - x30)$ hours of work = year.

The question now is, what number is x, when $x(365 - x30)$ is greatest. The number 5 is a common multiplier, therefore we have to consider $x73 - x^2 6$ only.

When x is	4,	$x73 - x^2 6$ is	196
,,	5,	,,	215
,,	6,	,,	222
,,	7,	,,	217

From which it appears that 6 is the number. Hence, 6 hours per day.

EXERCISE XV.

1. Given the length of the sidereal day 23h. 56m. 4·09s., and the length of the mean solar day 24h.; find the length of the year.

2. A person, on being asked what time it was, answered that the time past noon was three-fifths of the time till midnight. What was the time?

3. What time is it at Pekin, Calcutta, Rome, Washington, Sydney, when it is 6·30 p.m. at London (Greenwich)?

4. A fortress was originally provisioned for 60 days, but after 20 days 15,000 additional troops were driven to take shelter in it; in consequence of which the provisions held out for only 10 days subsequent thereto. What was the number of the original garrison?

5. If 24 boys or 15 men can do three quarters of a piece of work in $7\frac{1}{2}$ hours, in what time will 10 men and 12 boys do the remainder?

6. Two men A and B working together can do a piece of work in 10 days; but

if A stops working after 4 days, B can finish the work in 14 days more. Compare their rates of working.

7. A cistern is supplied from two taps, by one of which it can be filled in 39 minutes, and by the other in 52 minutes. In what time will it be filled by both together?

8. What time would 36 men, working $10\frac{1}{2}$ hours a day, require to build a wall which 24 men, working $9\frac{1}{3}$ hours a day, can build in 9 days?

9. A cistern is fitted with three pipes, one of which will fill it in 48 minutes, the second in an hour, and the third in half-an-hour: how long will it take to fill the cistern when all three pipes are open together?

10. Assume that 6 men can do as much work in an hour as 7 women, and 8 women as much as 11 boys, and that 5 men can do a certain piece of work in 10 hours: how long will it take 1 man, 2 women, and 3 boys together to do the same piece of work?

11. If B and C working together take p days to a piece of work, for which C and A together take q days, and A and B together take r days; find how long each would take by himself.

12. Assume that 4 English navvies can do as much work in a day as 5 French navvies, that 4 French navvies can do as much work as 7 negroes, and that 13 English and 12 French do a piece of work in 3 days: how long will it take 10 negroes to do that piece of work?

13. Compare the time of a place $7° 30' 15''$ west of Greenwich with Greenwich time.

14. Find the successive convergents to the difference between 365 days and the true solar year.

SECTION XVI.—SPEED.

Art. 106.—**Speed and Velocity distinguished.** It is important to distinguish between *speed* and *velocity*, or at least to discriminate between two different ideas, which these words may be used to fix. Velocity may be defined as rate of change of position with respect to time; while speed may be defined as the rate, with respect to time, of change of distance measured along a specified path. The elaborating of this distinction is due to Tait (Mechanics, *Ency. Brit.*, vol. xv., p. 681). Speed thus defined does not involve direction in its conception, while velocity does.

Both are expressed in terms of **L** per **T**; but, in the case of speed, **L** denotes a length merely; whereas, in the case of velocity, it denotes a vector (Art. 69). When our attention is restricted to motion along a definite path, it is not necessary to specify the direction of the velocity; it is sufficient to state whether it is backwards or forwards.

The idea which is reciprocal to that of speed is *slowness*. It is expressed in terms of **T** per **L**.

ART. 107.—**British Units.** According to the common usage of this country, we may have any of the units of length for **L**, and any of the units of time for **T**; for example, miles per hour, miles per minute, miles per second, yards per minute, feet per second, etc. The choice of each unit depends on the magnitude of the quantities of that kind which come into consideration. In the case of the motion of trains the distances coming into consideration are great, and so are the times occupied; hence the speed is commonly expressed in miles per hour. In the motion of a projectile the distance coming into consideration is not great, and the time occupied is small; hence foot per second is a more convenient unit.

Calculation, however, is usually facilitated by choosing one set of fundamental units. Hence in the British system of absolute units, the F.P.S. system, foot per second is the primary unit of speed.

ART. 108.—**Metric and C.G.S. Units.** The primary unit of speed in the French system is naturally the metre per second. However, as the first founders of the system departed from the metre in taking the centimetre to define the unit of mass (Art. 127), the founders of the C.G.S. system have adopted the centimetre throughout as the primary unit of length, and accordingly adopt the centimetre per second as the primary unit of speed.

As the foot per second and the centimetre per second involve

the same unit of time, the conversion of the former into the latter is the same as the conversion of the foot into the centimetre. The mean solar units being the same in all countries, the only conversions to which they give rise are those due to the relations of the several denominations of mean solar time to one another.

EXAMPLES.

Ex. 1. Express a speed of 60 miles per hour in terms of kilometres per second.

$$\begin{aligned} 1 \text{ kilometre} &= 1000 \text{ metres,} \\ 1 \text{ metre} &= 39{\cdot}37 \text{ inches,} \\ 36 \text{ inches} &= \text{yard,} \\ 1760 \text{ yards} &= \text{mile,} \\ 60 \text{ miles} &= \text{hour,} \\ 1 \text{ hour} &= 3{,}600 \text{ seconds;} \end{aligned}$$

$$\therefore \quad \frac{36 \times 1760 \times 60}{39370 \times 3600} \text{ kilometres} = \text{second,}$$

$$\textit{i.e.,} \quad \frac{176 \times 6}{39370} \text{ kilometres} = \text{second,}$$

$$\textit{i.e.,} \quad {\cdot}0268 \text{ kilometres} = \text{second.}$$

Ex. 2. Find the average speed of a lamplighter who spends 10 seconds at each lamp, and walks to the next, 25 yards off, at the rate of 5 miles an hour.

$$\begin{aligned} 10 \text{ secs. delay} &= \text{lamp,} \\ 1 \text{ lamp} &= 25 \text{ yards,} \\ 1760 \text{ yards} &= \text{mile,} \\ 5 \text{ miles} &= \text{hour walking;} \end{aligned}$$

$$\therefore \quad 2 \times 1760 \text{ secs. delay} = \text{hour walking,}$$

$$\textit{i.e.,} \quad \frac{2 \times 1730}{3600} \text{ hours delay} = \text{hour walking,}$$

$$\textit{i.e.,} \frac{44}{45} \text{ hours delay} = \text{hour walking.}$$

Hence the average speed is

$$5 \text{ miles} = 1 + \frac{44}{45} \text{ hour},$$

i.e., $$\frac{5 \times 55}{89} \text{ miles} = \text{hour},$$

i.e., $$2{\cdot}5 + \text{miles} = \text{hour}.$$

Ex. 3. A person standing on a railway platform noticed that a train took 21 seconds to pass completely through the station, which was 88 yards long, and that it took 9 seconds to pass himself. How long was the train, and at what rate per hour was it travelling?

Suppose x yards long.

Hence speed of train is

$$(88 + x) \text{ yds.} = 21 \text{ seconds};$$

but it is also

$$x \text{ yds.} = 9 \text{ secs.};$$

$$\therefore \quad 9(88 + x) = 21x,$$

$$\therefore \quad x = 66.$$

$$1 \text{ mile} = 1760 \text{ yds.},$$

$$66 \text{ yds.} = 9 \text{ secs.},$$

$$3600 \text{ secs.} = \text{hour};$$

$$\therefore \quad \frac{22 \times 120}{11 \times 16} \text{ miles} = \text{hour},$$

i.e., $$15 \text{ miles} = \text{hour}.$$

EXERCISE XVI.

1. Express a speed of 1 mile per hour in terms of feet per second; and of 1 foot per second in terms of mile per hour.

2. Express a speed of 60 miles per hour in terms of feet per second.

3. The speed with which light travels is 186,000 miles per second; express it in metres per second.

4. Reduce 1 kilometre per hour to centimetres per second.

5. Römer found that a ray of light took 16m. 36s. to cross the diameter of the earth's orbit. The mean distance of the sun from the earth is 92·39 million miles. Deduce the speed with which light travels, given that it travels uniformly.

6. Compare the speeds of two locomotives, one of which travels $397\frac{5}{6}$ miles in $11\frac{2}{5}$ hours, and the other $262\frac{4}{13}$ miles in $8\frac{4}{9}$ hours.

7. It is found on taking the log that a steamer goes a distance of six *knots*, each knot measuring 48 feet, while a sand-glass measures out 14 seconds. What is the speed of the ship in nautical miles per hour?

8. Express in miles per hour the difference of 60 kilometres per hour and 60 miles per hour.

9. A man walks at 5 kilometres per hour; express his speed in metres per second.

10. The speed of a ship is 12 knots per hour; express it in metres per second.

11. Express the following speeds in miles per hour—

Ordinary wind,	5 to 6	metres per second,
A fresh breeze,	10	,,
A tempest,	25 to 30	,,
A hurricane,	40	,,

12. Express the following speeds *first* in kilometres per second, *second* in centimetres per second.

Speed of a point on the equator of	Mercury	146·87	m. per sec.,
,, ,,	Mars,	244	,,
,, ,,	Venus,	454·58	,,
,, ,,	Earth,	463	,,
,, ,,	Sun,	2,048	,,

13. A tram-car goes round its circuit of $4\frac{1}{2}$ miles in one hour, stopping at two stations five minutes and two minutes respectively, and making twenty other stoppages of an average duration of 10 seconds each. Find the average speed of the car while in motion.

14. A local train makes its run of 13 miles and back once in every two hours, stopping half a minute at each of the fourteen stations on the line, and ten minutes at either terminus. What is its average speed when in motion?

15. An ordinary train takes ten hours to a certain journey, besides two hours in all of stoppages. The express goes 50 per cent. faster, and completes the journey in four hours less. What time does it lose in stoppages?

16. A man rides a certain distance and walks back in six hours; he could ride both ways in $3\frac{1}{2}$ hours; how long would it take him to walk both ways?

17. A person walked from Cambridge to Newmarket, a distance of 14 miles, and back in seven hours thirty minutes. His speed going to his speed returning was 8 to 7; find the speeds.

18. How long will it take a man to walk round a square field whose area is $5\frac{5}{8}$ acres, at the rate of one mile in $10\frac{2}{3}$ minutes?

19. A traveller has a hours at his disposal; he rides forward with a friend in a coach travelling b miles an hour, and has to return home at the end of his a hours on foot, walking c miles an hour. How far can he go with his friend?

20. A man starts to explore an unknown country, carrying provisions for ten days? He can walk 15 miles a day when carrying provisions for ten days, and he can go an extra mile a day for each day's provisions he gets rid of. What distance will he have walked by the time he has just exhausted his provisions?

21. A steamer made the passage from New York to Queenstown in 6 days 14 hours 18 minutes. The daily runs were 435, 410, 415, 433, 420, 426, and 275 knots. Find the average speed in knots per hour, and in statute miles per hour.

SECTION XVII.—RELATIVE SPEED.

ART. 109.—**Overtaking and Approach.** Suppose that A and B move with uniform speeds along the same path and in the same sense. Suppose that the speeds are

$$m\ \mathsf{L} \text{ by } A = \mathsf{T}, \tag{1}$$

and

$$n\ \mathsf{L} \text{ by } B = \mathsf{T}; \tag{2}$$

and suppose that m is greater than n. From these two rates we can deduce derived rates as in Art. 23.

From (1) and (2) we deduce

$$m\ \mathsf{L} \text{ by } A = n\ \mathsf{L} \text{ by } B. \tag{3}$$

By subtracting (2) from (1),

$$m - n\ \mathsf{L} \begin{array}{l}\text{gain by } A\\ \text{loss by } B\end{array} = \mathsf{T}. \tag{4}$$

By means of (1) and (2), (4) can be put into the forms

$$\frac{m-n}{m}\ \mathsf{L} \text{ gain by } A = \mathsf{L} \text{ by } A, \tag{5}$$

and

$$\frac{m-n}{n}\ \mathsf{L} \text{ gain by } A = \mathsf{L} \text{ by } B. \tag{6}$$

Suppose now that A and B move in opposite directions. By adding instead of subtracting we obtain

$$m + n\ \mathsf{L} \text{ approach by } A \text{ to } B = \mathsf{T}, \tag{7}$$

$$\frac{m-n}{m}\ \mathsf{L} \text{ approach} = \mathsf{L} \text{ by } A, \tag{8}$$

$$\frac{m-n}{n}\ \mathsf{L} \text{ approach} = \mathsf{L} \text{ by } B. \tag{9}$$

Art. 110.—**The Vernier.** The principle of the vernier is exactly similar. Suppose that the equivalences are

$$1 \text{ inch of scale} = \text{division},$$

and $$\cdot 9 \text{ inch of vernier} = \text{division};$$

then $$\cdot 1 \text{ inch gain by scale} = \text{division}.$$

If the scale overtakes the vernier after say 7 divisions, then the gain by the scale is ·7 inch; that is, the difference between the approximate and true reading is ·7 inch.

When the equivalence for the vernier is

$$1\cdot 1 \text{ inch of vernier} = \text{division},$$

then $$\cdot 1 \text{ inch gain by vernier} = \text{division}.$$

In this case the coincidence has to be sought for by going backwards instead of forwards. A vernier constructed according to the former plan is called a *sextant-vernier*, while one constructed according to the latter plan is called a *barometer-vernier*.

Examples.

Ex. 1. Two passenger trains having equal speeds, and consisting, the one of 12 carriages, the other of 14, are observed to take 10 seconds to pass one another. What is the speed, estimating the length of a carriage at 23 feet?

Let the speed of either train be

$$x \text{ ft.} = \text{sec.};$$

then their relative speed is

$$2x \text{ ft.} = \text{sec.}$$

$$10 \text{ sec.} \qquad \therefore\ 20x \text{ ft.}$$

But $$23 \text{ ft.} = \text{carriage},$$

$$12 + 14 \text{ carriages}, \qquad \therefore\ 26 \times 23 \text{ ft.}$$

Hence $$20x = 26 \times 23,$$

$\therefore$ $$x = 29\cdot 9.$$

Ex. 2. In running a race one mile long A beats B by 100 yards, and B beats C by 90 yards; by how many will A beat C?

$$1760 \text{ yards by } A = 1660 \text{ yards by } B,$$

1760 yards by B = 1670 yards by C,
1760 yards by A;

$\therefore$ $\dfrac{166 \times 1670}{176}$ yards by C,

i.e., $1575\frac{5}{44}$ yards by C;

$\therefore$ $1760 - 1575\frac{5}{44}$ yards by which A beats C,

i.e., $184\frac{39}{44}$ yards by which A beats C.

Ex. 3. A racecourse is 3,000 ft. long; A gives B a start of 50 ft. and loses the race by a certain number of seconds; if the course had been 6,000 ft. long, and they had both kept up the same speed as in the actual race, A would have won by the same number of seconds. Compare A's speed with B's.

a ft. by A = sec.,
b ft. by B = sec.;

$\therefore$ a ft. by A = b ft. by B.

3,000 ft. by A, $\therefore$ $\dfrac{3000}{a}$ secs.,

2,950 ft. by B, $\therefore$ $\dfrac{2950}{b}$ secs.

Hence the time by which A lost is

$$\frac{3000}{a} - \frac{2950}{b} \text{ secs.}$$

Similarly in the other case the time by which A wins is

$$\frac{6000}{a} - \frac{5950}{b} \text{ secs.}$$

Now it is given that

$$\frac{6000}{a} - \frac{5950}{b} = \frac{3000}{a} - \frac{2950}{b},$$

$$\therefore \quad \frac{a}{b} = \frac{90}{89}.$$

Hence 90 ft. by A = 89 ft. by B.

Ex. 4. A man walks, at three miles an hour, along a tram-line, and during his walk he is overtaken by 6 tram-cars. If the cars

start simultaneously at equal intervals of time, from both ends, and travel at the rate of 5 miles an hour, determine the number of tram-cars he should have met in the same time.

Speed of man, 3 miles = hour,
Speed of cars, 5 miles = hour;
$\therefore$ 8 miles approach = hour,
and 2 miles overtake = hour.

Since the distance between car and car is uniform
8 cars meet = 2 cars which overtake,
6 cars overtake;
$\therefore$ 24 cars meet.

Answer—24 cars.

EXERCISE XVII.

1. A traveller starts from A towards B at 12 o'clock, and another starts at the same time from B towards A. They meet at 2 o'clock at 24 miles from A, and the one arrives at A while the other is still 20 miles from B. What is the distance from A to B?

2. A steamer leaves Liverpool for New York, and a vessel leaves New York for Liverpool at the same time; they meet, and when the steamer reaches New York the vessel has as far to go as the steamer had when they met. If Liverpool be 3,000 miles from New York, how far out from Liverpool was the steamer when they met?

3. A starts to walk from Edinburgh to Glasgow, and at the same time B starts to walk from Glasgow to Edinburgh. A reaches Glasgow 9 hours after meeting B, and B reaches Edinburgh 6 hours 15 minutes after meeting A. Find in what time each has performed the journey.

4. A, who walks at the speed of $3\frac{3}{4}$ miles per hour, starts 18 minutes before B; at what speed must B walk to overtake A at the ninth milestone?

5. A messenger starts on an errand at the rate of 4 miles an hour; another is sent $1\frac{1}{2}$ hours after to overtake him; the latter walks at the rate of $4\frac{3}{4}$ miles an hour. When and where will he overtake him?

6. A passenger train going 41 miles an hour, and 431 feet long, overtakes a goods train on a parallel line of rails. The goods train is going 28 miles an hour, and is 713 feet long. How long does the passenger train take to pass the other?

7. A and B run a race a mile long, and A beats B by 100 yards; A then runs with C, and beats him by 200 yards; finally B runs the course with C. By how much does B beat C?

8. A and B run a race; B has 50 yards start, but A runs 20 yards while B runs 19. What must be the length of the course that A may come in a yard ahead of B?

9. In a 100 yards race A beats B by 5 yards and C by 10 yards. By how many yards does B beat C?

10. In a 100 yards race A can beat B by 10 yards; B in the same distance beats C by 10 yards. By how many will A beat C?

11. In a mile race A can beat B by 17 yards or $2\frac{5}{6}$ seconds. Find A's time over the course.

12. In a mile race A gives B 50 yards; B passes the winning post 5 minutes after the start; A passes it 5 seconds later. Which would win in an even race, and by what distance?

13. In a mile race between A and B, whose relative speeds are as 4 to 3, B had the start by 3 minutes, but was beaten by 80 yards. Required the speed of each in yards per minute.

14. A starts from a railway station, walking at the rate of 5 miles an hour; at the end of an hour B starts walking 4 miles an hour; at the end of another hour a train starts and passes A 25 minutes after it passes B. Find the speed of the train.

15. A tourist, having remained behind his companions, wishes to rejoin them on the following day. He knows they are 5 miles ahead, will start in the morning at 8 o'clock, and will walk at the rate of $3\frac{1}{4}$ miles an hour. When must he start in order to overtake them at 1 o'clock P.M., walking at the rate of 4 miles an hour, and resting once for half an hour on the road?

16. One man setting out from A travels towards B at the rate of 6 miles per hour; 2 hours afterwards a second man starts from A, and going 10 miles per hour reaches B 4 hours before the first man. Find the distance between A and B.

17. Two men start from a town on the same road—the first on foot, walking 18 miles in 7 hours; the second on horseback $5\frac{2}{3}$ hours later, walking 36 miles in 5 hours. Find the time in which the second gets (1) half as far, and (2) twice as far as the first.

18. Suppose that cars move on a tram-route at the average rate of 6 miles per hour, and are despatched from either end at intervals of 5 minutes, and that a man walks along the route at the rate of 4 miles an hour. How many cars per hour will meet him, and how many cars per hour will overtake him?

19. A man walking at 4 miles an hour along a tram-route observes that in the course of an hour he meets 20 cars, and is overtaken by 4. What is the average speed of the cars, and what is the average distance between two successive cars?

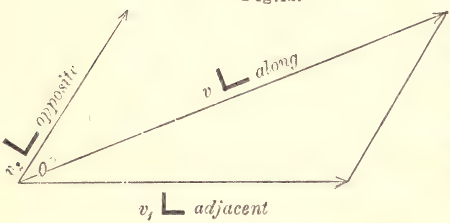
20. Show how to construct a vernier to make barometer readings to ·002 of an inch, when the divisions on the scale are twentieths of an inch.

21. The circumference of the limb of an angular instrument is divided into 1,080 parts; show how to construct a vernier which will read to a minute.

SECTION XVIII.—VELOCITY.

ART. 111.—**Velocity in one Plane.** Velocity, being a vector rate, can be resolved into components in the same manner as a vector. Suppose that our attention is restricted to one plane. The equivalence between the full velocity and its components is expressed by (Fig. 12)

Fig.12.



$$v \;\mathsf{L}\text{ along per }\mathsf{T} = v_1 \;\mathsf{L}\text{ adj. per }\mathsf{T} + v_2 \;\mathsf{L}\text{ opp. per }\mathsf{T}.$$

From this complete equivalence certain partial equivalences may be derived, as in Art. 25,

$$v_1 \;\mathsf{L}\text{ adj. per }\mathsf{T} = v \;\mathsf{L}\text{ along per }\mathsf{T},$$

or

$$\frac{v_1}{v} \;\mathsf{L}\text{ adj.} = \mathsf{L}\text{ along.}$$

This last, when the components are at right angles to one another, is the cosine of the direction of the velocity.

Similarly, under the same condition,

$$\frac{v_2}{v} \;\mathsf{L}\text{ opp.} = \mathsf{L}\text{ along}$$

is the sine of that direction. Also

$$v_2 \;\mathsf{L}\text{ opp. per }\mathsf{T} = v_1 \;\mathsf{L}\text{ adj. per }\mathsf{T};$$

$$\therefore \qquad \frac{v_2}{v_1} \;\mathsf{L}\text{ opp.} = \mathsf{L}\text{ adj.};$$

which, when the components are rectangular, is the tangent of the direction of the velocity.

The relation between the numbers is the same as that for the resultant and components of a simple vector. When the components are inclined at an angle $\theta°$,

$$v^2 = v_1^2 + v_2^2 + 2v_1v_2 \cos\theta.$$

When θ is 90 this becomes

$$v^2 = v_1^2 + v_2^2.$$

ART. 112.—**Velocity in Space.** When space is considered there are three rectangular components, as for example,

$$v_1 \text{ L east per T},$$
$$v_2 \text{ L north per T},$$
$$v_3 \text{ L up per T}.$$

The resultant speed is

$$\sqrt{v_1^2 + v_2^2 + v_3^2} \text{ L per T};$$

and the direction is fully specified by the three direction-cosines

$$\frac{v_1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \text{ L east } = \text{L along},$$

$$\frac{v_2}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \text{ L north} = \text{L along},$$

$$\frac{v_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \text{ L up } = \text{L along}.$$

EXAMPLES.

Ex. 1. While a train is moving with a velocity of 20 miles an hour alongside a station platform, the guard throws out a parcel with a horizontal velocity of 16·9 feet per second in a direction at right angles to the motion of the train. What will be the velocity of the parcel at the beginning?

First of all we have to see that the two components are expressed in terms of the same unit, say, feet per second.

$$1 \text{ hour} = 3{,}600 \text{ seconds},$$
$$20 \text{ miles} = \text{hour},$$
$$1760 \times 3 \text{ feet} = \text{mile};$$
$$\therefore \quad \frac{176}{6} \text{ feet} = \text{second},$$
$$i.e., \quad \tfrac{88}{3} \text{ feet} = \text{second}.$$

Thus, the rectangular components are

$$\tfrac{88}{3} \text{ feet parallel} = \text{second},$$
$$16{\cdot}9 \text{ feet across} = \text{second};$$

$$\therefore \quad \frac{16 \cdot 9 \times 3}{88} \text{ feet across} = \text{foot parallel,}$$

$$\textit{i.e.,} \quad \cdot 577 \quad ,, \quad = \quad ,,$$

Hence the tangent of the angle between the direction of motion of the parcel and the direction of motion of the train is ·577;

$$\therefore \quad \text{the angle is } 30°.$$

Also, since the components are rectangular,

$$\sqrt{(\tfrac{88}{3})^2 + (16 \cdot 9)^2} \text{ feet resultant} = \text{second,}$$

$$\textit{i.e.,} \quad 33 \cdot 8 \quad ,, \quad = \quad ,,$$

Ex. 2. A ship sailing due *N.* at the rate of 7 knots an hour is carried to the *E.* by a tide current of 4 knots an hour. Find her real velocity over the ground in knots correct to two places of decimals.

$$7 \text{ knots } N. = \text{hour,}$$

$$4 \text{ knots } E. = \text{hour;}$$

$$\therefore \quad \sqrt{49+16} \text{ knots} = \text{hour,}$$

$$\textit{i.e.,} \quad \sqrt{65} \quad ,, \quad = \quad ,,$$

$$\textit{i.e.,} \quad 8 \cdot 06 \quad ,, \quad = \quad ,,$$

Also, $\quad \frac{7}{4} \text{ knots } N. = \text{knot } E.,$

$$\textit{i.e.,} \quad 1 \cdot 75 \quad ,, \quad = \quad ,,$$

$$\therefore \quad \text{angle is about } 60°.$$

EXERCISE XVIII.

1. A steamer goes 9·6 miles per hour in still water. How long will it take to run 10 miles up a stream and return, the velocity of the stream being 2 miles an hour?

2. A party row down a river in three hours, and up in seven, their rate in still water being 5 miles an hour. Required the distance, and the velocity of the water.

3. Suppose that a steam-tug travels 10 miles an hour in still water when alone, but draws a barge 4 miles an hour. It has to take a barge 10 miles up a stream which runs 1 mile an hour, and then to return without the barge. How long will it take for the journey?

4. A vessel makes two runs on a measured mile, one with the tide in a minutes, and the other against the tide in b minutes. Find the velocity of the vessel through the water and of the tide, supposing both to be uniform.

5. A particle receives simultaneously three velocities, viz., 60 feet per second *N.*, 88 feet per second *W.* 30° *S.*, and 60 feet per second *E.* 30° *S.* Give the magnitude and direction of the resultant velocity.

6. A ship sailing due north at the rate of 8 knots per hour is carried to the east by a tide current of 4 knots an hour. Find her real motion over the ground in knots per hour correct to two places of decimals.

7. A river one mile broad is running downwards at the rate of 4 miles an hour, and a steamer moving at the rate of 8 miles an hour wishes to go straight across. How long will the steamer take to perform the journey, and in what direction must she be steered?

8. A boat is rowed in the direction of right across a river with a velocity of 8 miles an hour. The river has a velocity of 2 miles an hour, and a breadth of 800 feet. Find how far the boat will be carried down by the time it reaches the opposite bank.

9. A ship is sailing *ESE.* at the rate of 10 knots an hour, and the wind seems to blow from the *NW.* with a velocity of 6 knots per hour. Find the true velocity of the wind.

10. If a steamer have a velocity of 14 knots an hour due west, and the wind blows with a velocity of 7 knots an hour from the north; what will be the apparent velocity of the wind to one on board the steamer?

SECTION XIX.—ANGULAR VELOCITY.

ART. 113.—**Speed of Turning.** When a rigid body rotates round an axis, each point in the body has a speed proportional to its perpendicular distance from the axis. Hence speed of rotation or turning is expressed in the form

$$\omega \mathsf{L} \text{ arc per } \mathsf{L} \text{ radius per } \mathsf{T}.$$

Speed of turning may also be expressed in the form

$$n \text{ revolutions} \div \mathsf{T}.$$

The reciprocal idea is that of *periodic time*,

$$1/n \; \mathsf{T} \text{ per revolution}.$$

ART. 114.—**Angular Velocity and Moment of Velocity.** When a point moves in a plane, the rate of change of direction of the line joining it with a fixed point in the plane is called its angular velocity with respect to that point. It is specified, like

speed of turning, in the form

$$\omega \text{ radian} = \mathsf{T},$$

or

$$\omega\ \mathsf{L} \text{ arc per } \mathsf{T} = \mathsf{L} \text{ radius}.$$

Consider the motion in one plane of a point P round a fixed point O (Fig. 13). The moment of the velocity of P round O is proportional to the velocity of P, and to the perpendicular OM from O upon the direction of the velocity. Let the velocity be v L along per T, and the perpendicular be p L perpendicular. Then the moment of velocity is

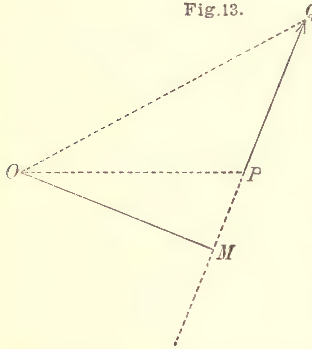
pv L along per T by L perpendicular,

or

pv L perpendicular by L along per T,

or pv L^2 area per T.

Fig. 13.



This is double the rate at which the radius-vector describes area, because a small sector traced out by the radius-vector is a triangle, not a parallelogram.

Examples.

Ex. 1. The speed of the periphery of a mill-wheel 12 feet in diameter is 6 feet per sec.; how many revolutions does the wheel make per minute?

$$6 \text{ feet arc.} = \text{sec},$$
$$1 \text{ rev.} = 12\pi \text{ feet arc},$$
$$60 \text{ sec.} = \text{min}.$$
$$\therefore\quad \frac{30}{\pi} \text{ rev.} = \text{min}.$$

Take $\pi = \frac{22}{7}$; then $9{\cdot}5$ rev. = min.

Ex. 2. A person inquiring the time of day is told that it is between V. and VI., and that the hour and minute hands of the clock are together. What o'clock is it?

$$\tfrac{1}{12} \text{ rev. by hour hand} = \text{hour},$$

$$1 \text{ rev. by minute hand} = \text{hour};$$

$$\therefore \quad 1 - \tfrac{1}{12} \text{ rev. gained by min. hand} = \text{hour},$$

$$\tfrac{5}{12} \text{ rev. to be gained};$$

$$\therefore \quad \tfrac{5}{12} \times \frac{1}{1 - \frac{1}{12}} \text{ hour required},$$

$$\textit{i.e.}, \quad \tfrac{5}{11} \text{ hour required.}$$

Hence the time is 5h. 27m. 16s.

Ex. 3. Find the average angular speed of the extremity of the minute hand of a watch which is three quarters of an inch in length.

$$2\pi \text{ inch arc} = \text{inch radius by hour},$$

$$\tfrac{3}{4} \text{ inch radius}; \quad \therefore \quad \frac{3\pi}{2} \text{ inch arc} = \text{hour}.$$

$$\text{But } 1 \text{ hour} = 3600 \text{ sec.},$$

$$\therefore \quad \frac{\pi}{2400} \text{ inch arc} = \text{sec.},$$

$$\textit{i.e.}, \quad \cdot 0013 \text{ inch arc} = \text{sec.}$$

Ex. 4. Find the number of revolutions per mile made by a wheel of $4\frac{1}{2}$ feet diameter.

$$\tfrac{9}{2} \text{ feet diam.}$$

$$\pi \text{ feet arc} = \text{foot diam},$$

$$\therefore \quad \frac{\pi \times 9}{2} \text{ feet arc.}$$

$$1 \text{ rev.} = \pi \tfrac{9}{2} \text{ feet arc},$$

$$3 \times 1760 \text{ feet arc} = \text{mile};$$

$$\therefore \quad \frac{1760 \times 2}{3\pi} \text{ revs.} = \text{mile},$$

$$\textit{i.e.}, \quad 373 \text{ revs.} = \text{mile}.$$

Ex. 5. Express in degrees and in circular measure the angle made by the hands of a clock at 3·35 o'clock.

$$30 \text{ degrees by hour hand} = \text{hour},$$

$$\therefore \quad \tfrac{1}{2} \text{ degree by hour hand} = \text{minute},$$

$$35 \text{ min.}; \qquad \therefore \quad \tfrac{35}{2} \text{ degs. past III.}$$

The minute hand is 4×30 degs. past III.,

$\therefore$ the angle between is $120 - 17{\cdot}5$ degs.,

i.e., $102{\cdot}5$ deg.

Now $3{\cdot}1416$ radians $= 180$ degs.,

$$\therefore \quad \frac{102{\cdot}5 \times 3{\cdot}1416}{180} \text{ radians,}$$

i.e., $1{\cdot}79$ radians.

EXERCISE XIX.

1. Find the multiplier for changing revolutions per minute into radians per second.

2. How soon after VIII. are the hour and minute hands directly opposite to each other?

3. Express in degrees, grades, and radians the angle made by the hands of a watch at 3·30 o'clock.

4. The minute and second hands of a watch point in the same direction at XII. When do they next point in the same direction?

5. At what time after n o'clock is the minute hand first ten minutes before the hour hand? What is the greatest value of n to allow this to happen within the hour?

6. Two clocks are together at XII.; when the first comes to I. it has lost a second; and when the second comes to I. it has gained a second. How far are they apart in 12 hours?

7. Two clocks are correct at mid-day; when the first clock indicates VI. in the afternoon, the second wants a minute to VI.; and when the second indicates midnight, the true time is 2 minutes past XII. What does the first clock indicate at midnight?

8. Two men walk opposite ways round a circular course. They meet for the first time at the north point, the sixth time at the east point. Where will they meet for the sixteenth time—and what are their relative speeds?

9. A clock loses at the rate of $8{\cdot}5''$ per hour when the fire is alight, and gains at the rate of $5{\cdot}1''$ per hour when the fire is not burning; but on the whole it neither loses nor gains. How long in the 24 hours is the fire burning?

10. In going 120 yards the forewheel of a carriage makes six revolutions more than the hindwheel. If each circumference were a yard longer, it would make only four revolutions more. Find the circumference of each wheel.

11. Two men start together to walk round a circular course, one taking 75 minutes to the round, the other 90. When will they be together again at the starting point?

12. The hour hand of a watch is 3/7 of an inch long, the minute hand 4/5 of an inch, and the second hand 1/3 of an inch. Compare the linear speeds of their points.

13. Deduce the equivalent of longitude for one minute of time, and for one second of time.

14. What is the circumferential speed of a wheel 28 feet in diameter when making five revolutions per minute?

15. The diameter of the earth is nearly 8,000 miles; required, the circumference of the earth at the equator, and the number of miles per hour which the inhabitants of latitude 60° are carried by the earth's diurnal rotation.

16. What is the velocity due to the earth's rotation of a person dwelling on the 45th degree of latitude?

17. The front wheel of a bicycle is 52 inches in diameter, and performs 5,040 revolutions in a journey of 65 minutes. Find the speed in miles per hour at which it has travelled, assuming the ratio of the circumference to the diameter of a circle to be as 22 to 7.

18. When a steamer sails due west, at the latitude of 45°, at the rate of 14 knots per hour, what is the rate at which the clock gains time?

19. A reaping machine works round a rectangular field of grain 357 yards by 216 yards at the average speed of 3 miles an hour, the breadth cut by the reaper being 5 feet. How long will it take to cut down the field?

20. Find the distance traversed in ploughing 12 acres of land when the furrow is cut 11 inches broad; also the time required when the horses move at the average speed of 2 miles an hour.

SECTION XX.—RATE OF CHANGE OF SPEED.

ART. 115.—**General Unit.** By *acceleration* is meant rate of change of velocity with respect to time. To each of the varieties of velocity discussed there is a corresponding variety of acceleration. Rate of change of speed and rate of change of velocity are both expressed in terms of

$$(\mathsf{L} \text{ per } \mathsf{T}) \text{ added per } \mathsf{T},$$

the only difference being that in the former case we have not, and in the latter we have to consider the direction of L.

A peculiarity of this rate is that time enters twice as the independent quantity. In order to be independent of one another, the latter interval must elapse before the former.

The unit of the British absolute system is the foot per second per second; and that of the C.G.S. system is the centimetre per second per second.

ART. 116.—**Rate of Change of Speed.** Here our attention is entirely restricted to one path. If the increment to the speed has the same sense as the existing speed, it is said to be an acceleration proper; if it has the opposite sense, it is said to be a retardation.

Suppose that the point is at one instant moving with a speed of v_1 **L** per **T**, and that after an interval of n **T** it is moving with a speed of v_2 **L** per **T**; then the change of speed in the course of the interval is $v_2 - v_1$ **L** per **T**. If this change has been made uniformly, then the rate of change during the interval is

$$\frac{v_2 - v_1}{n} \textbf{ L per T per T.}$$

ART. 117.—**Application of Rate to find Space passed over.** Given that a point is subject to a constant rate of change of speed,

a **L** per **T** per **T**,

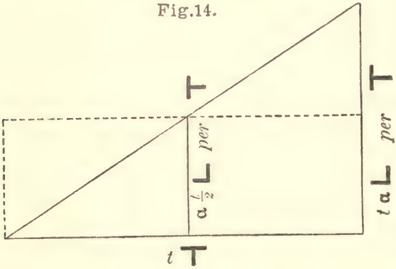
if the interval during which it goes on is t **T**, then the change of speed is ta **L** per **T** If the subsequent interval is t' **T**, then $t'ta$ **L** is the distance gone over due to the speed which was imparted before the interval began. Hence the rate may be viewed in the form of an equivalence

a **L** = **T** subs. by **T** prec.,

or a **L** per **T** subs. = **T** prec.

But when the intervals are simultaneous, the speed is 0 **L** per **T** at the beginning, and ta **L** per **T** at the end; and it increases uniformly during the interval. Hence (Art. 81), we shall get the correct result by supposing the average speed to have existed throughout the interval (Fig. 14). The

Fig.14.



average speed is $\frac{0+at}{2}$ L per T, *i.e.*, $\frac{at}{2}$ L per T.

Hence, instead of the previous equivalence, we have

$$\tfrac{1}{2}a \text{ L} = \text{T}^2.$$

We have T^2 because the two intervals are identical. The $\frac{1}{2}$ comes in for the same reason that it appears in the equivalence for the area of a triangle.

ART. 118.—**Derived Rate.** Given that a point is subject to a constant rate of change of speed

$$a \text{ L per T} = \text{T}, \tag{1}$$

then, as we have shown, the ratio of space described due to that acceleration is

$$\tfrac{1}{2}a \text{ L} = \text{T}^2. \tag{2}$$

By squaring the first equivalence we obtain

$$a^2(\text{L per T})^2 = \text{T}^2; \tag{3}$$

and by eliminating T^2 between (3) and (2), we deduce

$$a^2(\text{L per T})^2 = \tfrac{1}{2}a \text{ L},$$

or
$$2a(\text{L per T})^2 = \text{L}. \tag{4}$$

We shall afterwards find (Art. 159) that $2(\text{L per T})^2$ is the unit for expressing the kinetic energy of unit mass.

ART. 119.—**Composition of Effects.** Let the initial distance be s L, the initial speed v L per T, and the constant rate of change of speed a L per T per T.

As all the quantities are supposed to be along the same line, the composition is effected by simple addition; but the quantities may differ in sign. Hence, after t T, the velocity will be
$$v + at \text{ L per T};$$
and the distance will be

$$s + vt + \tfrac{1}{2}at^2 \text{ L}.$$

EXAMPLES.

Ex. 1. Assuming 32·2 as the foot-second measure of the acceleration produced by gravity, express the same quantity numerically in terms of the mile-hour unit.

$$1 \text{ mile} = 3 \times 11 \times 160 \text{ ft.},$$

$$32{\cdot}2 \text{ ft.} = \text{sec. by sec.},$$

$$1 \text{ sec. by sec.} = \frac{1}{3600} \times \frac{1}{3600} \text{ hour by hour};$$

$$\therefore \quad \frac{32{\cdot}2 \times 3600 \times 3600}{3 \times 11 \times 160} \text{ mile} = \text{hour by hour},$$

$$\textit{i.e.,} \quad \frac{322 \times 180 \times 15}{11} \text{ mile} = \text{hour by hour},$$

i.e., 79,036 miles per hour per hour.

Ex. 2. Recently in the case of a girl who threw herself from the top of the column in the Place Vendôme in Paris (height 40 metres), it was discussed whether death would be caused by the mere speed attained before reaching the base. What was the speed?

$$9{\cdot}8 \text{ metres per sec. per sec.},$$

$$\therefore \quad \frac{9{\cdot}8}{2} \text{ metres} = \text{sec.}^2,$$

$$40 \text{ metres};$$

$$\therefore \quad \frac{40}{4{\cdot}9} \text{ sec.}^2,$$

$$\textit{i.e.,} \quad \frac{400}{49} \text{ sec.}^2,$$

$$\textit{i.e.,} \quad \frac{20}{7} \text{ sec.}$$

Now $\quad 9{\cdot}8$ metres per sec. = sec.,

$$\therefore \quad \frac{20 \times 9{\cdot}8}{7} \text{ metres per sec.}$$

i.e., 28 metres per sec.

Or we can proceed thus

$$9{\cdot}8 \text{ metres per sec.} = \text{sec. of fall},$$

$$4{\cdot}9 \text{ metres fall} = (\text{sec. of fall})^2;$$

$$\therefore \quad (9{\cdot}8)^2(\text{metres per sec.})^2 = 4{\cdot}9 \text{ metres of fall},$$

$$40 \text{ metres of fall};$$

$$\therefore \quad 9{\cdot}8 \times 2 \times 40 \text{ (metres per sec.)}^2;$$

$$\therefore \quad \sqrt{49 \times 4 \times 4} \text{ metres per sec.},$$

i.e., 28 metres per sec.

Ex. 3. A balloon is 400 feet from the ground, and ascending at the rate of 10 feet per second. What time would a sandbag take to fall to the ground from it?

The sandbag has a velocity of 10 feet upwards per second, and when let fall it is subject to an acceleration of

32 ft. downwards per sec. per sec.;

hence $\frac{32}{2}$ ft. downwards $=$ (sec. of fall)^2.

Suppose t seconds taken to reach the ground;

then $t10$ ft. up, and $t^2\frac{32}{2}$ ft. down.

Now $t^2\frac{32}{2} - t10$ ft. $= 400$ ft.,

hence the equation $t^2 8 - t5 - 200 = 0$;

$$\therefore \quad t = \frac{5 \pm \sqrt{25 + 6400}}{16},$$

$$= \frac{5 \pm 80}{16} \text{ nearly},$$

$$= 5\tfrac{1}{3} \text{ nearly}.$$

Answer—$5\frac{1}{3}$ secs.

Ex. 4. St. Rollox stalk is 445 ft. high; at what rate must a bullet be shot vertically upwards to reach the top (disregarding the resistance of the air) in a second? What would be its speed when passing the top? How high would it rise?

1st. Let the velocity of projection be

v ft. up $=$ sec.,

$\therefore$ for 1 sec., v ft. up.

But gravity is acting downwards at

16 ft. $=$ sec.^{2};

$\therefore$ for 1 sec., 16 ft. down.

Hence $v - 16$ ft. $= 445$ ft.,

$\therefore$ $v = 461$.

2nd. The original velocity is 461 ft. up per sec.; gravity acts at 32 ft. down per sec. $=$ sec.,

$\therefore$ for 1 sec., 32 ft. down per sec.;

hence the velocity when passing the top is

$$461 - 32 \text{ ft. up per sec.},$$

i.e., 429 ft. up per sec.

3rd. 2×32 (ft. per sec.)^2 deducted = ft. rise,

$(461)^2$ (ft. per sec.)^2;

$$\therefore \quad \frac{(461)^2}{64} \text{ ft. rise,}$$

i.e., 3321 ft. rise.

Ex. 5. The speed of a railway train increases uniformly for the first three minutes after starting, and during this time it travels one mile. What speed, in miles per hour, has it now gained, and what space did it describe in the first two minutes?

$$1 \text{ mile} = (\tfrac{1}{20} \text{ hour})^2,$$

$\therefore$ 400 miles = hour^2,

$\therefore$ 800 miles per hour = hour,

$\frac{1}{20}$ hour,

$\therefore$ 40 miles per hour.

Again, 400 miles = hour^2,

$\frac{1}{30}$ hour;

$\therefore$ $\frac{400}{900}$ mile,

i.e., $\frac{4}{9}$ mile.

EXERCISE XX.

1. Express an acceleration of 500 centimetres per second per second in terms of the kilometre and minute.

2. The velocity and the acceleration of a moving point at a certain moment are both measured by 10 the foot and the second being the units of space and time. Find the numbers measuring them when the yard and the minute are the units.

3. On what arbitrary units does the numerical quantity g depend? If each unit becomes m times its former amount, what will be the new value of g?

4. Express 32·2 feet per second per second in terms of yard per minute per minute.

5. To ascertain the height of a precipice a stone was dropped from the edge and was observed to take three and a half seconds to reach the bottom. What is the height of the precipice?

6. A stone dropped from the top of a cliff is observed to reach the bottom in $6\frac{1}{4}$ seconds. Find the height.

7. A boy throws a stone vertically into the air with a velocity of 80 feet per second. How much time has he to escape from it returning?

8. A stone is let fall, and another is at the same instant projected upwards from a point 500 feet lower in the same vertical. With what speed must it be projected so that the two may meet half way?

9. How far has a body fallen from rest when it has acquired a velocity of (1) 20 feet per second, (2) 100 feet per second?

10. A stone, dropped from rest, falls under the action of gravity 65 feet during a particular second of time. How long before the end of this second did it begin to fall?

11. Two particles are let fall, the one from 100 feet, the other from 225 feet high, and they reach the ground at the same time. Find the interval between their times of starting.

12. A particle is dropped from a height; supposing it reaches the ground in 12 seconds, how much did it fall during the last second, and how far has it fallen altogether?

13. A stone is thrown downwards, and its average velocity for the second second of its fall is $2\frac{1}{2}$ times that for the first second. What was the initial velocity?

14. A rifle bullet is shot vertically downwards from a balloon at rest at the rate of 400 feet per second. How many feet will it pass through in two seconds, and what will be its velocity at the end of that time, neglecting the resistance of the air and estimating the acceleration due to gravity at 32?

15. A stone thrown vertically upwards strikes the ground after an interval of 10 seconds. With what velocity was it projected, and to what height did it rise?

16. If a body is projected upwards with a velocity of 120 feet per second, what is the greatest height to which it will rise, and when will it be moving with a velocity of 40 feet per second?

17. With what velocity must an arrow be shot vertically upwards in order that it may just reach the top of a stalk 150 feet high?

18. My watch beats five times each second. A boy throws a stone into the air vertically upwards, and I reckon $27\frac{1}{2}$ beats of my watch from the instant the stone leaves the boy's hand until it strikes the ground. Taking $g = 32$, show that the boy's hand when the stone left it was moving with a velocity of 88 feet per second; and find how high the stone went.

19. What is meant when it is said that the acceleration of the speed of a particle is 10, the units being foot and second? If the particle were moving at any instant at the rate of $7\frac{1}{2}$ feet per second, after what time would its speed be quadrupled? and what distance would it describe in that time?

20. A body describes distances of 120 yards, 228 yards, 336 yards, in successive tenths of a minute. Show that this is consistent with constant acceleration of its velocity, and find the numerical value of the acceleration if the units of time and distance are a minute and a yard.

21. A body is moving in a straight line with a uniform velocity of 10 feet per second. Suddenly a force begins to act upon it in a direction contrary to that of its motion, whose acceleration is 5 feet per second per second. In what sense and with what velocity will the body be moving at the end of $2\frac{1}{2}$ seconds from the moment the accelerating force began to act?

22. A particle is found to be moving in a straight line at the rate of 5 feet per second, a quarter of a minute afterwards at the rate of 50 feet per second, half a minute afterwards at 95 feet per second. Show that this is consistent with a constant rate of change of speed, and find its value.

SECTION XXI.—ACCELERATION.

ART. 120.—**Rate of Change of Velocity.** Acceleration being a vector quantity, is resolved and compounded in the same manner as a simple vector or as a velocity. When our attention is restricted to one plane, and to rectangular components, the equivalence between the acceleration and its components is expressed by a**L** along per **T** per **T** = a_1**L** adj. per **T** per **T** + a_2**L** opp. per **T** per **T**. When the components are rectangular, the numbers a, a_1, a_2, are connected by the condition

$$a^2 = a_1^2 + a_2^2.$$

From this equivalence partial equivalences may be derived, as in Art. 111.

The cosine of the direction of the acceleration is given by

$$\frac{a_1}{\sqrt{a_1^2 + a_2^2}} \text{ L adj.} = \text{L along},$$

and the tangent by

$$\frac{a_2}{a_1} \text{ L opp.} = \text{L adj.}$$

ART. 121.—**Rate of Deviation.** One mode in which rate of change of velocity can be resolved into rectangular components is by taking the direction of motion for the time being as one line, and the perpendicular to it as the other line. The former com-

ponent is the rate of change of speed, which has been already considered; there remains for consideration the component along the transverse line. It does not affect the speed, but it alters the direction of the motion. It is proportional to the square of the velocity of the point and to the curvature produced. The dependence is fully expressed by

$$1\ \mathsf{L}\text{ per }\mathsf{T}\text{ per }\mathsf{T} = (\mathsf{L}\text{ per }\mathsf{T})^2\text{ by (radian per }\mathsf{L}\text{ arc).}$$

By transforming the right hand unit we obtain equivalent forms,

$$\begin{aligned}1\ \mathsf{L}\text{ per }\mathsf{T}\text{ per }\mathsf{T} &= (\mathsf{L}\text{ per }\mathsf{T})^2\text{ per }\mathsf{L}\text{ radius (Art. 74),}\\ &= (\mathsf{L}\text{ arc per }\mathsf{L}\text{ radius per }\mathsf{T})\text{ by }(\mathsf{L}\text{ per }\mathsf{T}),\\ &= (\text{radian per }\mathsf{T})\text{ by }(\mathsf{L}\text{ per }\mathsf{T}).\end{aligned}$$

ART. 122.—**Dimensions.** The dimensions of the several units expressed above are said to be the same. The dimension of a unit with respect to a fundamental unit as **L** is reckoned by taking the number of times it enters directly and the number of times it enters inversely and taking the difference. In transforming from one set of fundamental units to another it is this difference upon which the transformation depends. The dimensions of **V** are 3 with respect to **L**; of **L** per **T**, 1 with respect to **L**, and -1 with respect to **T**; of **L** per **T** per **T**, 1 with respect to **L**, and -2 with respect to **T**.

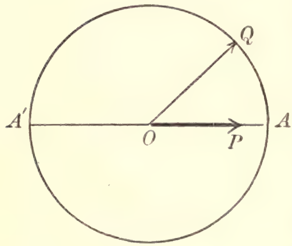
ART. 123.—**Simple Harmonic Motion.** Let a point move round a circle with uniform angular velocity, then the component of this motion along any diameter of the circle is a simple harmonic motion.

Let the point Q move round the circle O, whose radius is a **L**, with a uniform angular velocity ω radian per **T**. Let AA' be the line of the simple harmonic motion, and suppose that t **T** have elapsed since Q was at A.

First, the position of P. The point Q will then be at an angle of ωt radians, and the component along AA' of the vector to Q will be $a\cos(\omega t)$ **L** (Fig. 15).

Second, the velocity of P. The linear velocity of Q will then be $a\omega$ **L** per **T** in the direction of the tangent at Q. The velocity of P is the component in the direction OA; hence it is

Fig.15.

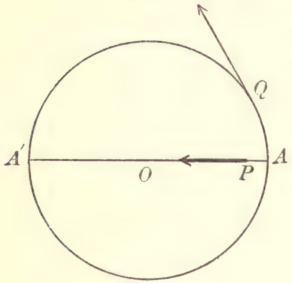


$$-a\omega \sin(\omega t) \text{ L per T}.$$

It has the negative sign because it is towards O (Fig. 16).

Third, the acceleration of P. The acceleration of Q will be $a\omega^2$ **L** per **T** per **T** in the direction towards the centre O (Art. 121). Hence the acceleration of P, being the component along AA', will be

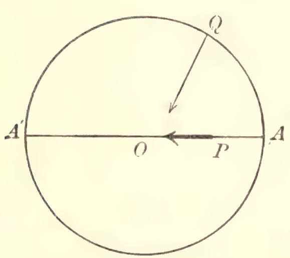
Fig.16.



$$-a\omega^2 \cos(\omega t) \text{ L per T per T}.$$

It has the negative sign because it is towards O (Fig. 17).

Fig.17.



Examples.

Ex. 1. A sphere of glass rolls down a smooth inclined plane, the inclination of which is 30°. Its velocity at a certain point is 70 cm. per sec., and at a second point below 140 cm. per sec. What is the distance between the two points?

980 cm. downwards = sec. by sec.,
sin 30°, *i.e.*, $\frac{1}{2}$ cm. along plane = cm. downwards,

$$\therefore \quad \frac{980}{2} \text{ cm. along plane per sec.} = \text{sec.}$$

Hence, $\frac{980}{4}$ cm. along plane = sec.^{2},

$\therefore$ 980 (cm. along plane per sec.)^2 = cm. along plane,
70 cm. per sec.

$$\therefore \quad \frac{70^2}{980} \text{ cm. along plane.}$$

Similarly, $$\frac{140^2}{980} \text{ cm. along plane.}$$

Hence, distance along plane is

$$\frac{3 \times 70^2}{980} \text{ cm.}$$

$$i.e., \quad 15 \text{ cm.}$$

Ex. 2. A heavy body on a level plane has simultaneously communicated to it an upward vertical velocity of 48 feet per second, and a horizontal velocity of 25 feet per second. Find its greatest height, its range, and its whole time of flight.

2×32 (foot per sec.)^2 deducted = foot rise,
48^2 (foot per sec.)^2,

$$\therefore \quad \frac{48^2}{64} \text{ feet rise,}$$

$$i.e., \ 36 \ \text{feet rise.}$$

Again, 48 feet upwards per sec.,
32 feet downwards per sec. = sec.,

$$\therefore \quad \frac{48}{32} \text{ secs. time of rise;}$$

$$\therefore \quad 2 \times \frac{48}{32} \text{ secs. time of flight,}$$

$$i.e., \quad 3 \text{ secs. time of flight.}$$

Again,
25 feet horizontal = sec.
3 sec. of flight, $\therefore$ 3×25 feet of range.
i.e., 75 feet of range.

Ex. 3. A tooth in the blade of a reaper describes a simple harmonic motion of one and a half inch amplitude in a period of one seventh of a second. What is its maximum velocity and its maximum acceleration?

Here the uniform angular speed is

$$2\pi \text{ inch arc per inch radius} = \tfrac{1}{7} \text{ sec.};$$

$$\therefore \quad 14\pi \text{ inch arc per sec} = \text{inch radius},$$

$$\tfrac{3}{2} \text{ inch radius},$$

$$\therefore \quad 21\pi \text{ inch arc per sec.}$$

Now, the simple harmonic component is greatest when it is equal to the circular velocity; hence its greatest value is

$$21\pi \text{ inch per sec.}$$

Similarly, the maximum acceleration is

$$294\pi^2 \text{ inch per sec. per sec.}$$

EXERCISE XXI.

1. A heavy body starting from rest slides down a smooth plane inclined 30 degrees to the horizon. How many seconds will it occupy in sliding 240 feet down the plane, and what will be its velocity after traversing this distance?

2. From a point in a smooth inclined plane a ball is rolled up the plane with a velocity of 16·1 feet per second. How far will it roll before it comes to rest, the inclination of the plane to the horizon being 30 degrees? Also, how far will the ball be from the starting-point after five seconds from the beginning of motion?

3. Two bodies start together from rest, and move in directions at right angles to each other. One moves uniformly with a velocity of 3 feet per second, the other moves under the action of a constant force. Determine the acceleration due to this force, if the bodies at the end of 4 seconds are 20 feet apart.

4. A stone is let fall from the top of a railway carriage which is travelling at the rate of 30 miles an hour. Find what horizontal distance and what vertical distance the stone will have passed through in one tenth of a second.

5. The time of flight of a bullet on a horizontal rifle-range is observed to be 5 seconds; find the greatest elevation it attained.

6. A body is projected horizontally from the top of a tower with a velocity of 100 feet per second; find its distance from the point of projection at the end of 2 seconds.

7. From the top of a tower 169 feet high, a ball is projected horizontally with a velocity of 100 feet per second. When will it reach the ground, and at what distance from the foot of the tower?

8. If a body is projected in a direction inclined to the horizon by 45 degrees, and strikes the horizontal plane passing through the point of projection after 5 seconds, what is the velocity of projection?

9. A ball is projected with a velocity of 60 feet per second at an elevation of 15 degrees to the horizon. What will be its range on a horizontal plane as compared

with the height ascended by a body projected vertically upwards with a velocity of 30 feet per second?

10. A rifle is pointed horizontally, with its barrel 5 feet above a lake. When discharged, the ball is found to strike the water 400 feet off. Find approximately the velocity of the ball.

11. A balloon is carried along at a height of 100 feet from the ground with a velocity of 40 miles an hour; a stone is dropped from it. Find the time before the stone reaches the ground, and the distance from the point where it reaches the ground to the point vertically below the point where it left the balloon.

12. What is the average velocity of a point executing a simple harmonic motion for the time occupied in moving from the one to the other extremity of its range, its maximum velocity being 5 feet per second?

13. A particle is describing simple harmonic motion in a period of 1/10 of a second, and with an amplitude of 4 centimetres. Find the acceleration of the particle when at the extremity of its range. Find also the velocity of the particle when passing through the middle of its range.

CHAPTER FOURTH.

DYNAMICAL.

SECTION XXII.—MASS.

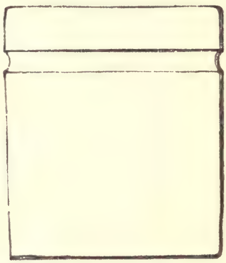
ART. 124.—**Mass and Weight.** By the mass of a body is meant the quantity of matter in it. Mass is the intrinsic property of a body; whereas weight is an accidental property depending on the presence of another body in the neighbourhood. These two ideas of mass and weight are confounded in the popular mind, and are not clearly discriminated in many text-books. A clear perception of the distinction greatly facilitates the application of arithmetic in the case of many problems.

Mass is a fundamental idea, and the general unit of mass is appropriately denoted by **M**.

ART. 125.—**Imperial Standard of Mass.** In the Imperial system the Standard of Mass is a cylinder of platinum, constructed in 1844, and now in the custody of the Warden of the Standards. It is denominated the "Imperial Pound Avoirdupois." The previous standard of mass, which was lost along with the standard of length, was a Pound Troy. The provision which had been made for its restoration was that one cubic inch of distilled water at 62° Fahr., the barometer standing at 30 inches, weighed 252·458 grains. This provision, however, was repealed on the recommendation of a scientific committee, and the new standard

was constructed from authentic copies of the old, the size of the standard being at the same time changed to the pound avoirdupois. Four parliamentary copies of the new standard pound were

Standard Troy Pound of 1758, to which the lost standard was similar.



Imperial Standard Pound—Height, 1·35 inch; Diameter, 1·15 inch.

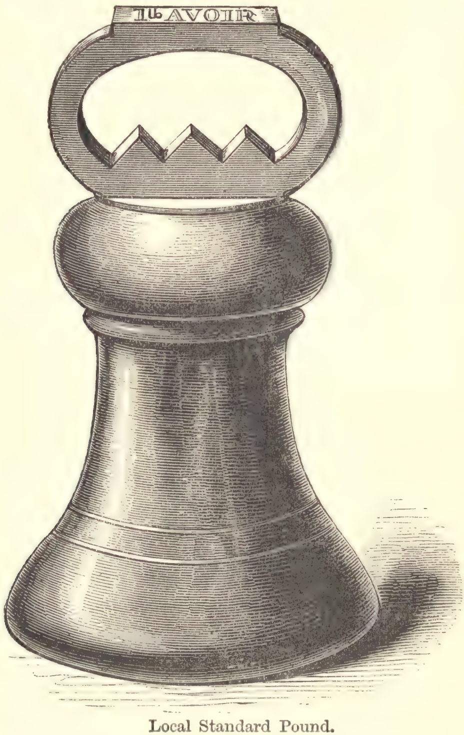
prepared, and deposited along with those of the yard. A fifth copy has been prepared to be used instead of the imperial pound in all ordinary comparisons.

According to the Weights and Measures Act, the comparison of any other mass with the standard of mass is to be done by weighing *in vacuo*.

Art. 126.—**Derived Units of Mass.** The other units of mass in the Imperial system which have a special name (see table appended), such as the ton, ounce, grain, are defined as a multiple or a sub-multiple of the pound. The avoirdupois denominations form the principal legal system; the ounce troy and its decimal

derivatives form the legal system for weighing bullion, and may be used in the sale of precious articles. For the retail trade in drugs, apothecaries' denominations may be used.

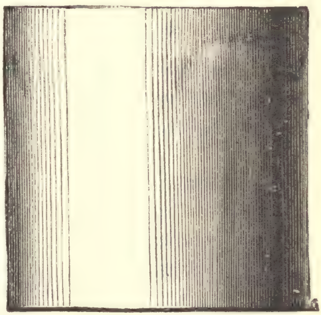
Local standard weights are constructed by the Standards Department, and distributed along with the local standard measures. The figure represents the form of the modern local standard avoirdupois weights.



Local Standard Pound.

Art. 127.—**Metric Standard of Mass.** In the Metric system the kilogramme was originally defined as the mass of a cubic decimetre of distilled water at the temperature of its maximum

density (about 4° C.). Distilled water is very suitable for a standard substance on account of its being obtainable everywhere in a state of purity, its homogeneity, and the invariability of its density at a given temperature. By weighing and measurement the mass contained in a cubic decimetre of water at its standard state was found, and a piece of platinum was constructed to represent that mass. The piece of platinum is called the *kilogramme des archives;* it, or rather a standard kilogramme subsequently constructed, is now the ultimate standard for the kilogramme in the same way as a piece of platinum is the ultimate standard for the pound. In the Standards Department there is a standard kilogramme of platinum, which is similar to the *kilogramme des archives.*



English Standard Platinum Kilogramme.

The gramme is the one-thousandth part of the kilogramme, and therefore the mass of a cubic centimetre of water at its temperature of maximum density.

In the C.G.S. system the gramme is chosen for the unit of mass.

IMPERIAL UNITS OF MASS (WEIGHTS).

AVOIRDUPOIS.

1 *grain*	= 1/7000 pound	= ·0648 gramme.
1 *dram*	= 1/256 pound.	
1 *ounce* = 16 drams	= 1/16 pound	= 28·35 grammes.
	1 POUND (lb.)	= 0·4536 kilogramme.
1 *stone*	= 14 pounds.	
1 *hundredweight* (cwt.)	= 112 pounds	= 50·80 kilogrammes.
1 *ton* = 20 cwt.	= 2240 pounds	= 1·016 millier.
1 *cental*	= 100 pounds.	

TROY.

1 *ounce troy* = 480 grains = 31·10 grammes.
1 "pennyweight" = 24 grains. 1 "pound troy" = 5,760 grains.

APOTHECARIES'.

1 "scruple" = 20 grains; 1 "drachm" = 3 scruples;
1 ounce troy = 8 drachms.

METRIC UNITS OF MASS (WEIGHTS).

1 millier or tonneau	= 1,000 kilogrammes.
1 quintal	= 100 kilogrammes.
1 myriagramme	= 10 kilogrammes.
	1 KILOGRAMME = 15,432·10 grains.
1 hectogramme	= ·1 kilogramme.
1 dekagramme	= ·01 kilogramme.
1 gramme	= ·001 kilogramme.
1 decigramme	= ·0001 kilogramme.
1 centigramme	= ·00001 kilogramme.
1 milligramme	= ·000001 kilogramme.

The authorized abbreviation for gramme is *g.*, but it is customary with English writers to use *gm.*, in order to distinguish between gramme and grain.

EXAMPLES.

Ex. 1. Find the rate connecting franc per kilogramme with pence per lb., when the course of exchange is 25 fr. 20 c. per £.

240 pence = 25·20 francs,

10,000 lb. = 4536 kilogrammes,

∴ dividing the one equivalence by the other,

$$\frac{24}{1{,}000} \text{ pence per lb.} = \frac{25\cdot20}{4536} \text{ francs per kgm.,}$$

i.e., $$\frac{24 \times 4536}{25200} \text{ pence per lb.} = \text{franc per kgm.}$$

i.e., 4·32 pence per lb. = franc per kgm

The reciprocal rate is

$$\frac{100}{432} \text{ franc per kgm.} = \text{penny per lb.},$$

$$\textit{i.e.,} \quad \cdot 231 \text{ franc per kgm.} = \text{penny per lb.}$$

Ex. 2. From the definitions of the gallon and of the litre deduce the relation of the former to the latter.

$$\begin{aligned} 1 \text{ gallon of water} &= 10 \text{ lb. of water}, \\ 1 \text{ lb. of water} &= 453\cdot 6 \text{ gms. of water}, \\ 1 \text{ gm. of water} &= \text{c.c. of water}, \\ 1{,}000 \text{ c.c. of water} &= \text{litre of water}; \\ \therefore \quad 1{,}000 \text{ gallon} &= 4{,}536 \text{ litres}, \\ \textit{i.e.,} \quad \cdot 22 \text{ gallon} &= \text{litre}. \end{aligned}$$

EXERCISE XXII.

1. Express a pound avoirdupois as the decimal of a pound troy, and an ounce avoirdupois as the decimal of an ounce troy.
2. Express 22 kilogrammes in pounds, and 25 pounds in kilogrammes.
3. Reduce 50 kilogrammes to hundredweights, and 1,000 kilogrammes to tons.
4. Express 4*d*. per lb. in terms of shilling per cwt. and pound per ton.
5. Reduce 1 franc per kilogramme to pence per lb. when the course of exchange is 25·10 francs = pound.
6. Reduce 100 francs per quintal to shillings per cwt. when the rate of exchange is 25·60 francs = pound.
7. Convert 5*s*. 6*d*. per lb. into francs per kilogramme when the course of exchange is 25 fr. 20 c. per pound.

SECTION XXIII.—DENSITY.

ART. 128.—**General Unit.** By the *density* of a substance is meant the rate connecting the mass with the volume. It is expressed in terms of **M** per **V**, the systematic unit being **M** per L^3. The reciprocal idea is **V** per **M**, which is denominated by Clerk-Maxwell the *rarity*,* and by Rankine the *bulkiness*.†

* *Heat*, p. 82. † *Rules and Tables*, p. 147.

ART. 129.—**British Units.** The British unit commonly used is pound per cubic foot. The value one pound per cubic foot does not express the density of any substance, because the pound was not defined directly by the cubic foot and a standard substance. The pound, however, originally had a relation to the density of water; for, taking the ounce its sixteenth part, the density of water is very approximately

1000 oz. per cubic foot,

which gives

62·5 lb. per cubic foot.

The true value for the temperature of maximum density is 62·425 lbs. per cubic foot.

ART. 130.—**Metric Units.** In the case of the metric units, 1,000,000 grammes per cubic metre expresses the density of water, the million being introduced because the kilogramme was defined by the cubic decimetre. Thus 1 gramme per cubic centimetre expresses the density of water, and this is the unit in terms of which density is expressed in the C.G.S. system.

The density of pure water at 4°C. is, more exactly, 1·000013 gm. per cc. (See Art. 151.)

ART. 131.—**Surface-density; Line-density.** Suppose that a body has a uniform density throughout $\rho \, \mathsf{M} = \mathsf{L}^3$. For a rectangular parallelepiped (Art. 93)

$$1 \; \mathsf{L}^3 = \mathsf{L} \text{ long by } \mathsf{L} \text{ broad by } \mathsf{L} \text{ thick};$$

therefore, for any parallepiped within the body,

$$\rho \; \mathsf{M} = \mathsf{L} \text{ long by } \mathsf{L} \text{ broad by } \mathsf{L} \text{ thick}.$$

If the body exist in the form of a plate of uniform thickness, then

$$1 \; \mathsf{L}^3 = \mathsf{L}^2 \text{ surface by } \mathsf{L} \text{ thick};$$

and $\therefore$ $\qquad \rho \; \mathsf{M} = \mathsf{L}^2 \text{ surface by } \mathsf{L} \text{ thick},$

or $\qquad \rho \; \mathsf{M} \text{ per } \mathsf{L}^2 \text{ surface} = \mathsf{L} \text{ thick}.$

Suppose that the thickness is d L, then the density is

$$d\rho \; \mathsf{M} \text{ per } \mathsf{L}^2 \text{ surface}.$$

This is called *surface-density*, and the letter used to denote its general value is σ.

If the body exist in the form of a rod or wire of uniform cross-section, then

$$1\ \mathsf{L}^3 = \mathsf{L}^2 \text{ cross-section by } \mathsf{L} \text{ long,}$$

and $\therefore$ $\rho\ \mathsf{M}$ per L long $= \mathsf{L}^2$ cross-section.

The value of the cross-section gives the value of the *line-density*. Let it be $a\ \mathsf{L}^2$, then

$$a\rho\ \mathsf{M} \text{ per } \mathsf{L} \text{ long.}$$

A letter sometimes used to denote the general value of M per L long is λ.

ART. 132.—**Rainfall.** In the case of rainfall we have to consider a horizontal sheet of water. The density of water is

$$62{\cdot}5 \text{ lb.} = \text{ft.}^3$$

Hence $\quad 62{\cdot}5$ lb. $=$ ft.^{2} surface by ft. deep,

or $\quad 5{\cdot}2$ lb. $=$ ft.^{2} surface by inch deep,

or $\quad 5{\cdot}2$ lb. per ft^2. surface $=$ inch deep.

By a number of inches of rainfall is meant the number of inches of depth. An inch of depth is equivalent to $5{\cdot}2$ lbs. of water per square foot of horizontal surface.

EXAMPLES.

Ex. 1. A cistern is $4\frac{1}{2}$ by $3\frac{1}{4}$ by $2\frac{1}{3}$ ft. What weight of water at $62\frac{1}{2}$ lbs. per cubic ft. can it hold?

$$\tfrac{9}{2} \text{ ft.} \times \tfrac{13}{4} \text{ ft.} \times \tfrac{7}{3} \text{ ft.}$$

$$1 \text{ cubic ft.} = \text{ft.} \times \text{ft.} \times \text{ft.},$$

$$\frac{125}{2} \text{lbs.} = \text{cubic ft.},$$

$$\therefore \quad \frac{9 \times 13 \times 7 \times 125}{2 \times 4 \times 3 \times 2} \text{ lbs.},$$

$$\textit{i.e.,} \quad \frac{34125}{16} \text{ lbs.},$$

$$\textit{i.e.,} \quad 2132{\cdot}8125 \text{ lbs.}$$

Ex. 2. Gold can be beaten out to leaf of the thickness of 1/3800 mm.; platinum can be made into wire 1/20000 mm. thick. What is the surface-density of the gold leaf, and the line-density of the platinum wire.

1st. 19·4 gm. = cubic cm.

= sq. cm. by cm. thick,

$\frac{1}{38000}$ cm. thick,

$\therefore \quad \frac{194}{380000}$ gm. = sq. cm.,

i.e., $\frac{194}{38}$ gm. = sq. metre,

i.e., 5·1 gm. = sq. metre.

2nd. 21 gm. = cub. cm.,

$\therefore$ 21 gm. per cm. long = sq. cm. cross-section,

$\frac{\pi}{4}$ sq. cm. = (cm. diam.)^2,

$\frac{1}{200000}$ cm. diam.,

$\therefore \quad \frac{21 \times \pi}{4 \times 4 \times 10^{10}}$ gm. per cm. long,

$\frac{21 \times 22}{16 \times 7}$ gm. = 10^{10} cm. long,

41 gm. = 10^{11} cm. long,

= 10^6 kilometres long.

EXERCISE XXIII.

1. A cubic foot of water weighs 1,000 oz. avoirdupois; required the relation of the kilogramme to the hundredweight.

2. A town of 241,000 inhabitants is supplied with water at the rate of 25 gallons per head per day; find the total supply both in volume and in mass for one year.

3. If the mass of a cubic inch of water be 252·5 grains, find the number of cubic inches in a ton of water.

4. The density of granite is 42 cwt. per cubic yard; what is it in lb. per cubic foot?

5. A ton of stone measures 13 cubic feet; what is the weight in kilogrammes of a cubic metre of the stone?

6. Reduce pound per cubic foot to gramme per cubic centimetre.

7. The density of water is 62·4 lb. per cubic foot; what is its bulkiness?

8. How many cubic feet of coal must be taken by a steamer going on a ten days' voyage, having engines of 1,000 horse-power? Rate of consumption of coal is 7 lbs. per horse-power per hour, and the bulkiness of the coal is 40 cubic feet per ton.

9. The density of granite is 160 lbs. per cubic foot; a paving block is 4 inch wide by 9 inch deep by 12 inch long. Find the number of tons required to pave a street one mile long and 20 yards broad, allowing an interval of ten per cent. between the blocks.

10. The average density of gunpowder is ·5268 oz. per cubic inch; find the amount required to fill a boring of $1\frac{1}{4}$ inch diameter and one yard long.

11. The density of water is ·036 lb. per cubic inch; express it in terms of ton per cubic foot, and of lb. per square inch per foot.

12. Given that the density of iron rail is 10·08 lb. per yard per square inch; find the mass required for 100 miles when the section of the rail is 6 square inches.

13. The population of Great Britain in 1881 was 35,262,762, and the area is 120,830 square miles; what is the average density of the population?

14. Given that the line density of a round bar of cast iron is 2·45 lb. per foot per inch diam. square, what is the weight of a pipe 2 yards long, having a bore of 16 inches, and a thickness of $\frac{3}{4}$ inch?

15. A flat bar of iron, $4\frac{3}{4}$ inch broad, and $\frac{5}{8}$ of an inch thick, has a linear density 9·91 lb. per foot; deduce the value of lb. per inch broad per eighth-inch thick per foot long.

16. The line-density of steel wire of $2\frac{1}{8}$ inch circumference is 4 lb. per fathom; what is the line-density of wire of the same material $2\frac{3}{4}$ inch in circumference?

17. Express the density of wrought iron in the form of, *first*, lbs. per square foot per inch thick; *second*, lbs. per linear yard per square inch section; *third*, lbs. per foot run per inch diameter square.

18. Given 10·6 bricks per square foot surface per brick thick; find the superficial area given by 10,000 bricks in a wall $2\frac{1}{2}$ bricks thick.

19. Transform the density of water given as 62·425 lbs. per cubic foot into the form suitable for calculations of rainfall, namely, pound per square mile per inch of rainfall, and pound per acre per inch of rainfall.

20. A cubic foot of copper, weighing 560 lbs., is rolled into a square bar 40 feet long. An exact cube is cut from the bar, what is its mass to four decimals of a pound?

21. Find the mass of zinc required to cover a rectangular roof 30 feet long by 20 feet broad, having a slope of 3 to 1, the superficial density of the covering being 7 cwt. per 100 square feet.

SECTION XXIV.—SPECIFIC MASS.

ART. 133.—**Relative Density.** Let the densities of two substances A and B be

$$m\ \mathsf{M} \text{ of } A = \mathsf{V},$$

and

$$n\ \mathsf{M} \text{ of } B = \mathsf{V}.$$

We deduce

$$m\ \mathsf{M} \text{ of } A = n\ \mathsf{M} \text{ of } B,$$

or

$$m/n\ \mathsf{M} \text{ of } A = \mathsf{M} \text{ of } B.$$

This rate expresses the density of A relatively to that of B, or the number of units of mass of A which are equivalent in volume to one unit of mass of B.

Different substances may thus be compared with one standard substance. Solids and liquids are compared with water; gases with air or with hydrogen.

For example,

$$11{\cdot}4\ \mathsf{M} \text{ of lead} = \mathsf{M} \text{ of water},$$
$$13{\cdot}596\ \mathsf{M} \text{ of mercury} = \mathsf{M} \text{ of water},$$
$$15{\cdot}96\ \mathsf{M} \text{ of oxygen} = \mathsf{M} \text{ of hydrogen}.$$

Equivalences are of two kinds—absolute and relative. Density is an example of the former kind, and relative density of the latter kind.

ART. 134.—**Specific Mass and Specific Gravity.** The density of a substance relatively to that of a standard substance is properly called *Specific Mass.* It is usually called *Specific Gravity*, as a consequence of not distinguishing weight from mass. These ideas are different; their numerical values, however, are the same, because the weight of a body is proportional to its mass, and is independent of its physical constitution.

By taking the reciprocals

$$1/m\ \mathsf{V}\ \text{of}\ A = \mathsf{M},$$
$$1/n\ \mathsf{V}\ \text{of}\ B = \mathsf{M},$$

we deduce
$$n/m\ \mathsf{V}\ \text{of}\ A = \mathsf{V}\ \text{of}\ B.$$

This expresses the *relative bulkiness* which is the reciprocal of the relative density.

SPECIFIC MASS AND DENSITY.

	SUBSTANCE.	M OF SUBSTANCE = M OF WATER OR GM. PER C.C.	LB. OF SUBSTANCE PER CUBIC FOOT.
Solid Metals.	Aluminium,	2·6	160
	Antimony,	6·7	420
	Bismuth,	9·8	613
	Copper,	8·9	537
	Gold,	19·4	1208
	Iron, cast,	7·3	451
	Iron, wrought,	7·8	486
	Lead,	11·4	709
	Platinum,	21·5	1344
	Silver,	10·4	654
	Steel,	7·8	487
	Tin,	7·3	455
	Zinc,	7·2	437
Alloys.	Bell metal,	8·05	503
	Brass,	8·4	524
	Gold, standard,	17·724	1106·42
	Silver, standard,	10·312	643·72
	Speculum metal,	7·4	465
Woods.	Ash,	·7	45
	Beech,	·7	43
	Cork,	·24	15
	Ironwood,	1·15	71
	Lignum-vitæ,	1·3	83
	Oak,	·8	48
	Pine, yellow,	·5	32
	Brick, ordinary,	2·2	137
	Glass, flint,	3·0	187
	Glass, crown,	2·5	156
	Granite,	2·7	168
	Ivory,	1·8	114
	Marble,	2·8	175
	Quartz,	2·65	165
	Sandstone,	2·3	144

SPECIFIC MASS AND DENSITY.—*Continued.*

	SUBSTANCE.	M OF SUBSTANCE = M OF WATER OR G.M. PER C.C.	LB. OF SUBSTANCE PER CUBIC FOOT.
Liquids at 32° Fahr., but water at 39°·1 Fahr.	Water, pure,	1	62·425
	Water, sea,	1·026	64
	Alcohol, pure,	·791	49
	Ether,	·716	45
	Hydrochloric acid,	1·2	75
	Mercury,	13·596	848·75
	Nitric acid,	1·2	75
	Oil, linseed,	·94	59
	Oil, olive,	·915	57
	Oil, whale,	·923	58
	Sulphuric acid,	1·84	115
Gases at 32° Fahr. and under one atmosphere of pressure.	Air,	001293	·0807
	Carbonic acid,	·00197	·1234
	Hydrogen,	·0000895	·0056
	Oxygen,	·00143	·0893
	Nitrogen,	·00125	·0786
	Steam (ideal),	·00080	·0502

EXAMPLES.

Ex. 1. A cubic foot of fresh water weighs 62·4 lbs., find in cubic feet the space occupied by one ton of sea water.

62·4 lbs. of fresh water = cubic foot,

1·026 lbs. of salt water = lb. of fresh water,

112 × 20 lbs. of sea water;

$$\therefore \quad \frac{112 \times 20}{62{\cdot}4 \times 1{\cdot}026} \text{ cubic feet,}$$

$$i.e., \quad \frac{700{,}000}{20{,}007} \text{ cubic feet,}$$

i.e., 35 cubic feet very nearly.

Ex. 2. Sodium has to alcohol the relative density 1·23, and alcohol has to water the relative density ·79; what is the density of sodium relatively to water?

$$1{\cdot}23\ \mathsf{M}\ \text{sodium} = \mathsf{M}\ \text{alcohol},$$
$$\cdot 79\ \mathsf{M}\ \text{alcohol} = \mathsf{M}\ \text{water},$$
$$\therefore\quad 1{\cdot}23 \times {\cdot}79\ \mathsf{M}\ \text{sodium} = \mathsf{M}\ \text{water},$$
$$\text{i.e.,}\quad {\cdot}97\ \mathsf{M}\ \text{sodium} = \mathsf{M}\ \text{water}.$$

Observation—In order that two relative equivalences may be combined in this manner, they must of course be equivalences in the same respect; and the conclusion is an equivalence in the same respect.

Ex. 3. Eleven cubic inches of iron weigh as much as seven cubic inches of lead, and the price per ton of lead is £15, of iron £4. The value of a certain block of lead is 36*l.* 17*s.* 11*d.*, what would be the value of a block of iron of the same size?

£36 17 11 of lead

$$1 \text{ ton of lead} = 15£ \text{ of lead},$$
$$7 \text{ tons of iron} = 11 \text{ tons of lead},$$
$$4£ \text{ of iron} = \text{ton of iron};$$
$$\therefore\quad 36\ \ 17\ \ 11 \times \frac{7 \times 4}{15 \times 11}\ £ \text{ of iron}.$$
$$\text{i.e.,}\quad 3\ \ 7\ \ 1 \times \frac{28}{15} \qquad \text{,,}$$
$$\text{i.e.,}\quad £6\ \ 5\ \ 2\tfrac{1}{2}$$

Ex. 4. The specific gravity of gold is 19·3, that of silver is 10·4. What is the composition of an alloy of gold and silver whose specific gravity is 17·6, no change of volume being supposed to accompany the combination of the metals?

Let the composition by volume of the alloy be

$$a\ \mathsf{V}\ \text{gold} + b\ \mathsf{V}\ \text{silver} = a + b\ \mathsf{V}\ \text{alloy},$$
$$19{\cdot}3\ \mathsf{M}\ \text{per}\ \mathsf{V}, \quad 10{\cdot}4\ \mathsf{M}\ \text{per}\ \mathsf{V};$$

therefore the composition by mass is

$$19{\cdot}3\ a\ \mathsf{M}\ \text{gold} + 10{\cdot}4\ b\ \mathsf{M}\ \text{silver} = 19{\cdot}3\ a + 10{\cdot}4\ b\ \mathsf{M}\ \text{alloy}.$$

Hence the density of the alloy is

$$\frac{19{\cdot}3\ a + 10{\cdot}4\ b}{a+b}\ \mathsf{M}\ \text{per}\ \mathsf{V}.$$

Hence, $$\frac{19\cdot3\ a+10\cdot4\ b}{a+b}=17\cdot6,$$

from which $$a/b=72/17.$$

EXERCISES XXIV.

1. A gallon of fresh water measures 277·271 cubic inches, and contains 10 lbs. avoirdupois. A ton of sea water measures 35 cubic feet. What is the mass of a gallon of sea water in pounds and decimals?

2. If 100 cubic inches of oxygen, under certain circumstances of pressure and temperature, contains 35 grains, and a cubic inch of mercury contains 0·49 lbs., how many cubic inches of the oxygen would contain the same quantity of matter as a cubic inch of mercury?

3. Of two bodies one has a volume of 5 cubic inches, the other of one-fifth of a cubic foot; the mass of the former is 15 oz., and of the latter 12·8 lb. What is the ratio of the density of the first to that of the second?

4. A flask holds 27 oz. of water. What mass will it hold of an oil whose specific mass is 0·95?

5. A cubic foot of water contains 1,000 ounces. 502·5 ounces of lead of specific gravity 11·5, and 440 ounces of iron of specific gravity 8, are placed in a cistern of the capacity of one cubic foot. Find the quantity of water necessary to fill the cistern.

6. From the relative densities to water of zinc, iron, tin, copper, lead, find the relative densities to iron of each of the other four metals.

7. The line-density of iron wire of No. 10 Birmingham wire gauge is 4·96 lb. per 100 lineal feet; what is the line-density of copper wire and of brass wire of the same gauge?

8. Sodium has to alcohol the relative density 1·23, and water has to alcohol the relative density 1·26; what is the density of sodium relatively to water?

9. If the specific gravity of a specimen of milk be m, and that of pure milk s; calculate the proportion of water added.

10. What must be the volume of a mass of wood of relative density 0·5, in order that when it is attached to 500 gms. of iron of relative density 7, the mean density of the whole may be equal to that of water?

11. If the price of whisky, the specific gravity of which is ·75, be 16*s.* a gallon, find the price when it is mixed with water so as to have the specific gravity ·8.

12. A Prussian dollar, made of an alloy of silver and copper, has the specific gravity 10·05. Determine the relative amount of silver and of copper in it, the specific gravities of these metals being 10·5 and 8·7 respectively.

13. A nugget of gold mixed with quartz weighs 10 oz. The specific gravity of gold is 19·35, of the quartz 2·15, and of the nugget 6·45. Find the mass of the

gold and of the quartz contained in the nugget; find also the ratio of their volumes.

14. A mixture is made of 7 cubic centimetres of sulphuric acid (specific gravity, 1·843) and 3 cubic centimetres of distilled water; and its specific gravity when cold is found to be 1·615. Determine the contraction which has taken place.

15. The density of a mixture of two liquids being supposed to be an arithmetical mean between those of the components; determine the ratio of the volumes of the components contained in the mixture.

16. Several liquids which do not alter their volume when mixed are shaken together; determine the specific gravity of the mixture from their specific gravities.

17. Half a pint of a liquid which is half as dense again as water is mixed with a pint of water; what is the density of the mixture?

18. A rod of uniform cross section 18 in. long weighs 3 oz.; its specific gravity is 8·8; what fraction of a square inch is the area of its cross section?

19. What is the mass of a cast-iron ball having a diameter of 6 inches; and of a cast-iron cylinder having the same diameter and 4 feet long?

20. A ditch 3 feet deep is dug round a square garden containing one tenth of an acre; find its width in order that the removed earth may raise the garden one foot.

21. Two liquids are mixed *first* by volume in the proportion of 1 to 4, and *second* by mass in the proportion of 4 to 1; the resulting specific masses are 2 and 3 respectively. Find the specific masses of the liquids.

SECTION XXV.—MASS-VECTOR.

ART. 135.—**Idea of Mass-Vector.** The ideas of dynamics differ from those of geometry and kinematics by the introduction of the idea of mass. From the idea of a vector we derive that of a *mass-vector*, which is proportional to a vector and to a mass. This term was introduced by Clerk-Maxwell,* and it is expressed in terms of **M** by **L**.

A mass-vector can be resolved and compounded in the same manner as a simple vector.

ART. 136.—**Centre of Mass.** The centre of mass (commonly

* *Matter and Motion*, p. 49.

called centre of gravity) of a number of material particles situated in a straight line is a point such that were the whole mass placed there, the value of **M** by **L** would be the same as before. The distance along the straight line from the origin to the centre of mass may be called the *equivalent* distance. (Compare Art. 31.)

When the particles are situated in one plane, then the centre of mass is a point which satisfies the above condition for two independent axes; and when they are in space, for three independent axes. The vector from the origin to the centre of mass may in a similar manner be called the equivalent-vector. The mass-vector due to the equivalent-vector and the whole mass is the resultant of the several component mass-vectors.

ART. 137. When a body of uniform density is symmetrical with respect to a plane, the centre of mass is somewhere in the plane of symmetry; when it is symmetrical with respect to two planes, the centre of mass lies in the axis of symmetry; and when it is symmetrical with respect to three planes, the centre of mass coincides with the centre of symmetry.

CENTRE OF MASS.

(The body being of uniform density.)

Triangle.—From a vertex along two thirds of the line to the middle point of the opposite side.

Semicircle.—From the vertex along ·5756 of the radius.

Pyramid or Cone.—From the apex along three fourths of the axis.

Hemisphere.—From the vertex along five eighths of the radius.

EXAMPLES

Ex. 1. At the corners of a cube weights are placed of 1, 2, 3, 4, 5, 6, 7, 8 lbs. respectively; determine their centre of mass.

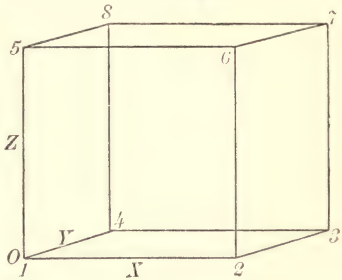
Let the side of the cube be **L** (Fig. 18.) Then for the direction of X we have

$(2\times 1+3\times 1+6\times 1+7\times 1)$ lb. by **L**,

i.e., 18 lb. by **L**;

but the whole mass is 36 lbs., therefore $\frac{18}{36}$ **L** is the distance of the centre of mass along the direction of X.

Fig. 18.



Similarly

$\frac{22}{36}$ **L** along Y, and $\frac{26}{36}$ **L** along Z.

Hence

$$\tfrac{1}{2}\,\mathbf{L} \text{ along } X,\ \tfrac{11}{18}\,\mathbf{L} \text{ along } Y,\ \tfrac{13}{18}\,\mathbf{L} \text{ along } Z.$$

The length of the vector from the corner 0 to the centre of mass is

$$\frac{\sqrt{9^2+11^2+13^2}}{18}\mathbf{L},\ i.e.,\ 1{\cdot}07\,\mathbf{L};$$

and its direction-cosines are

$$\frac{9}{19{\cdot}3}\mathbf{L} \text{ along } X = \mathbf{L} \text{ along vector, etc.}$$

Ex. 2. A uniform rod, 10 feet long, is bent at right angles at a point 4 feet from its end. Find the perpendicular distances of the centre of mass of the rod from the two straight portions of it.

Let the line-density of the rod, which is uniform, be denoted by 1 **M** per foot; then in the shorter piece there is 4 **M**, and in the longer piece 6 **M**. Since either piece is uniform and symmetrical, its centre of mass is at its mid-point.

We have now reduced the mass to two masses of 4 **M** and 6 **M** situated at the two mid-points. Their centre of mass is in the joining line, and at a distance of 6/10 of the line from the mid-point of the shorter piece.

The component of the vector from the corner to the centre of mass along the shorter piece is 4/5 feet, and the component along the longer piece is 9/5 feet.

EXERCISE XXV.

1. Find the centre of mass of two spheres of brass, of 1 inch and 2 inches diameter, placed at a distance of 5 inches, the distance being measured from the centres.

2. Where is the centre of mass of a square tin plate? If the plate weighs 5 oz. and a small body weighing 2 oz. is placed at one corner of the plate, where will the centre of mass of the whole be?

3. Find the centre of mass of the figure A when the pieces are of uniform material, and the central piece is half a side piece in length, and is joined at the mid-points of the sides.

4. Find the centre of mass of a T square, the two pieces being of the same material, and equal in length, breadth, and thickness.

5. Find the centre of mass in the case of a wooden F, the principal pieces being of the same length, and the central piece of half that length. Also for an E.

6. Find the centre of mass of the letter Y, the three pieces being uniform, and each one inch in length, and the two upper inclined at an angle of 60°.

7. A wooden vessel, 6 inches square and 6 inches in height, with a neck 2 inches square and 3 inches in height, is full of water. Find the position of the centre of mass of the water.

SECTION XXVI.—MOMENTUM.

ART. 138.—**Unit of Momentum.** The idea of momentum is derived from the idea of velocity by introducing the idea of mass. The momentum of a body is proportional to its mass and to its velocity; the general unit is M by (L per T). This unit is equivalent to (M by L) per T, when it is understood that the mass remains constant during change of time. Hence the bracket may be dispensed with, and either of these interpretations put upon

M by L per T.

Momentum is a directed quantity, its direction being the same as that of the velocity (or mass-vector) on which it depends. Hence it is resolved and compounded after the manner of directed quantities.

If the speed only of a body is considered, then we consider only its speed-momentum.

There is no special name for the unit of momentum in any of the systems of units. The unit may be denoted at length as lb. by ft. per sec., kilogramme by metre per sec., gm. by cm. per sec.

ART. 139.—**Impulse.** By an impulse is meant the cause which produces a change of momentum in a body, whether the change takes place in the magnitude or in the direction, or in both the magnitude and the direction. The second law of motion states that the impulse is measured by the change of momentum produced. Let I denote the systematic unit of impulse, then

$$1\,\mathsf{I} = \mathsf{M} \text{ by } \mathsf{L} \text{ per } \mathsf{T}.$$

ART. 140.—**Momentum per Volume and Current.** Liquid bodies, such as water and mercury, when at a uniform temperature throughout, are also uniformly dense. Suppose that the density of such a liquid is $\rho\,\mathsf{M}$ per L^3, and that its velocity is $v\,\mathsf{L}$ per T, then by multiplying together these rates we get

$$\rho v\,\mathsf{M} \text{ by } \mathsf{L} \text{ per } \mathsf{T} = \mathsf{L}^3;$$

that is, ρv units of momentum per unit of volume of the liquid.

Consider a cross-section perpendicular to the direction of the velocity. The density can be expressed as

$$\rho\,\mathsf{M} \text{ per } \mathsf{L}^2 \text{ cross-section per } \mathsf{L} \text{ normal},$$

and the velocity is

$$v\,\mathsf{L} \text{ normal per } \mathsf{T};$$

hence, by eliminating the common unit,

$$\rho v\,\mathsf{M} \text{ per } \mathsf{L}^2 \text{ cross-section per } \mathsf{T}.$$

This is the idea of *current per unit cross-section*, and it is ultimately equivalent to the idea of momentum per unit of volume. Let the cross-section be $a\,\mathsf{L}^2$, then $a\rho v\,\mathsf{M}$ per T. This is the idea of *current*.

EXAMPLE.

Ex. A shell of 60 lbs. weight is moving before explosion at the rate of 400 feet per second. In consequence of the explosion, a

piece weighing 14 lbs. is projected forward in the direction of motion with an additional velocity of 300 feet per second. What will now be the velocity of the remainder?

The mass of the piece is 14 lbs., and its additional velocity is 300 feet forward per second, therefore its additional momentum is

$$14 \times 300 \text{ lb. by foot forward per second.}$$

By the third law of motion the additional momentum of the remainder of the shell has the same magnitude but the opposite direction, therefore it is

$$14 \times 300 \text{ lb. by foot backward per second;}$$

but the mass of the remainder is $60 - 14$ lbs.;

$\therefore$ its additional velocity is

$$\frac{14 \times 300}{60 - 14} \text{ feet backward per second,}$$

i.e., 91·3 „

Hence the new velocity of the remainder is

$$400 - 91{\cdot}3 \text{ feet forward per second,}$$

i.e., 308·7 „

EXERCISE XXVI.

1. Of two bodies moving with constant velocities, one describes 36 miles in 1 h. 20 m., the other 55 feet in $1\frac{1}{4}$ sec.; the former weighs 50 lb., the latter 72 lb. Compare their momenta.

2. How far would a cannon-ball weighing 10 lb. travel in one minute, supposing it to possess the same momentum as a rifle bullet of 2 oz. moving with the velocity of 1,000 ft. per sec.

3. A man whose weight is 12 stones falls freely from a height of 64 feet. Calculate, neglecting the resistance of the air, the velocity and momentum acquired on reaching the ground.

4. A mass of snow, 28 lbs. in weight, falls from the roof of a house to the ground, a distance of 40 feet. Calculate the momentum.

5. A ball weighing 10 lbs. is projected vertically upwards with an initial velocity of 1,660 feet per second. Find its velocity and its momentum after 30 seconds and after 60 seconds.

6. Find the momentum per cubic foot in the case of a stream flowing at 2 miles an hour. Express in terms of the F.P.S. unit.

7. Compare the amounts of momentum in a pillow of 20 lbs. which has fallen through one foot vertically, and an ounce-bullet moving at 200 feet per second.

8. Calculate the momentum of a hammer of 5 tons, let fall half a foot.

9. A ball of 56 lb. is projected with a velocity of 1,000 feet per sec. from a gun weighing with the carriage 8 tons. Find the maximum velocity of recoil of the gun.

10. A pipe with a diameter of 2 inches delivers water at the rate of 9·8 gallons per minute; what is the velocity of the water.

SECTION XXVII.—FORCE.

ART. 141.—**General Unit.** Force is rate of impulse with respect to time, and is expressed in terms of **I** per **T**.

Since 1 **I** = **M** by **L** per **T**,

1 **I** per **T** = (**M** by **L** per **T**) per **T**;

that is, unit of force is equivalent to unit of momentum per unit of time. The rate-unit **I** per **T** is conveniently denoted by one letter **F**.

As the mass of the body considered is supposed not to change as time goes on, the above unit (**M** by **L** per **T**) per **T** is equivalent to **M** by (**L** per **T** per **T**); and thus the unit of force may be considered as derived from the unit of acceleration by introducing the unit of mass as a factor.

Another equivalent mode of viewing the unit is (**M** by **L**) per **T** per **T**.

Hence the unit may be written

M by **L** per **T** per **T**,

and any one of these interpretations may be given to it.

ART. 142.—**Special Units.** In the British system of scientific units the principal unit of force is the

pound by foot per second per second.

It is denominated the *poundal*, a term invented for the purpose by Professor James Thomson.

In the French system the unit principally used is the

kilogramme by metre per second per second.

In the C.G.S. system the unit chosen is

gramme by centimetre per second per second;

it is denominated the *dyne*.

The founders of the C.G.S. system have adopted the prefixes *mega* and *micro* to denote respectively a multiple and a sub-multiple of one million. Thus

1 megadyne = 1,000,000 dynes.

1,000,000 microdynes = 1 dyne.

Units of force such as the above are called absolute units, because they are defined entirely in terms of the fundamental units of length, mass, and time.

ART. 143.—**Intensity of a Force; Inertia.** Since

1 **F** = **M** by **L** per **T** per **T**,

1 **F** per **M** = **L** per **T** per **T**,

and 1 **F** per (**L** per **T** per **T**) = **M**.

By **F** per **M** is expressed the *intensity of a force;* it is equivalent to the acceleration.

By **F** per (**L** per **T** per **T**) is expressed *inertia;* it is equivalent to the mass.

ART. 144.—**Gravitation Measure of Force.** In practice it is very convenient to measure a force by comparing it with the weights of known masses at the place. Let the acceleration produced by gravity at the place be

32·2 foot per second per second;

then, since 1 poundal per lb. = foot per second per second,

we have 32·2 poundals per lb.

If the counterbalancing mass is m lbs., the force is

m 32·2 poundals.

When the mass is half an ounce m is $\frac{1}{32}$, hence one poundal is nearly equivalent to the weight of a half ounce.

Similarly, if the intensity of gravity at the place be given as

981 cm. per second per second,

then, since 1 dyne per gm. = cm. per second per second,

we have 981 dynes per gm.

If the counterbalancing mass is m gm., the force is

m 981 dynes.

If the value of **M** is given, say m **M**, but not that of **L** per **T** per **T**, then all that we know is expressed by

m **F** = **L** per **T** per **T** ;

so that the force is not given absolutely, but only relatively to the constant intensity of gravity at the place.

In Britain the standard intensity of gravity is the intensity at the latitude of London at the level of the sea, namely,

32·187 feet per second per second.

In France, it is the intensity at the latitude of Paris at the level of the sea, namely,

9·8087 metre per second per second.

Examples.

Ex. 1. A force acts on a mass of 8 oz. for 6·9125 mins., and produces a velocity of 10 feet per sec. Express the magnitude of the force in poundals and dynes.

10 feet per sec. per 6·9125 min.

$\therefore \quad \dfrac{1}{6 \times 6{\cdot}9125}$ feet per sec. per sec ; and $\frac{1}{2}$ lb.

Now 1 poundal = lb. by foot per sec. per sec.

$\therefore \quad \dfrac{1}{2 \times 6 \times 6{\cdot}9125}$ poundal,

i.e., ·012 + „

Again, $\dfrac{1}{2 \times 6 \times 6{\cdot}9125}$ lb. by foot per sec. per sec.;

453·6 gm. = lb.,

30·48 cm. = ft.,

$\therefore$ by substitution,

$$\frac{453{\cdot}6 \times 30{\cdot}48}{2 \times 6 \times 6{\cdot}9125} \text{ gm. by cm. per sec. per sec.,}$$

$$i.e., \quad 166 + \text{dyne.}$$

Ex. 2. What force, expressed in pounds weight, will in a minute give a mass of one ton a velocity of 10 miles per hour?

The mass is 112×20 lb.,

The additional velocity is $\dfrac{10 \times 1760 \times 3}{60 \times 60}$ feet per sec.,

$$i.e., \quad \frac{44}{3} \quad \text{,,}$$

$\therefore$ the change of momentum is $\dfrac{112 \times 20 \times 44}{3}$ lb. by ft. per sec.

Now, this is given uniformly in 60 seconds,

$$\therefore \quad \frac{112 \times 20 \times 44}{3 \times 60} \text{ lb. by foot per sec. per sec.,}$$

$$i.e., \quad \frac{112 \times 44}{9} \text{ poundals.}$$

If the intensity of gravity at the place is $32{\cdot}2$ feet per sec. per sec., then

$$32{\cdot}2 \text{ poundals} = \text{lb.,}$$

$$\therefore \quad \frac{112 \times 44}{9 \times 32{\cdot}2} \text{ lbs.,}$$

$$i.e., \quad 17 + \text{lbs.}$$

Ex. 3. A body under the action of a constant force traverses in the tenth second from rest twice the distance it would have traversed in the fifth second from rest under the action of gravity. Compare the forces.

Suppose the intensity of the force to be x poundals per lb., then the acceleration it produces is x feet per sec. per sec, therefore $x/2$ feet per sec^2., therefore the distance traversed during the tenth second is $(10^2 - 9^2)x/2$ feet, *i.e.*, $19x/2$ feet.

Suppose that the intensity of gravity at the place is $32{\cdot}2$ poundals per lb., then it may be shown in the same manner that the

distance traversed during the fifth second from rest is $(5^2 - 4^2)32{\cdot}2/2$ feet. Now it is given that

$$\frac{19}{2}x = 2\,\frac{9 \times 32{\cdot}2}{2},$$

$$\therefore \quad x = 30{\cdot}5.$$

The intensity of the force is 30·5 poundals per lb.

Ex. 4. A mass of 2 lbs. is drawn along a smooth horizontal table by a mass of 1 lb. hanging vertically; required the space described in 4 seconds.

The force acting is $1 \times 32{\cdot}2$ poundals, and the mass moved is 3 lb., therefore the intensity is

$$\frac{1 \times 32{\cdot}2}{3} \text{ poundals per lb.}$$

But

$$1 \text{ poundal per lb.} = \text{ft. per sec. per sec.},$$

$$\therefore \quad \frac{1 \times 32{\cdot}2}{3} \text{ ft. per sec. per sec.}$$

$$\therefore \quad \frac{32{\cdot}2}{2 \times 3} \text{ ft.} = \text{sec.}^2,$$

$$4 \text{ sec.}, \quad \therefore \frac{4^2 \times 32{\cdot}2}{2 \times 3} \text{ ft.},$$

$$\textit{i.e.}, \quad 85{\cdot}9 - \text{ft.}$$

Ex. 5. A mass of 488 grammes is fastened to one end of a cord which passes over a smooth pulley. What mass must be attached to the other end in order that the 488 grammes may rise through a height of 200 centimetres in 10 seconds, the intensity of gravity at the place being 980 dynes per gramme?

Suppose x gms.; then, as the intensity of gravity is 980 dynes per gm., the force acting is

$$(x \times 980 - 488 \times 980) \text{ dynes},$$

$$\textit{i.e.}, \quad (x - 488)980 \text{ gm. by cm. per sec. per sec.}$$

The mass moved is $x + 488$ gms., therefore the acceleration is

$$\frac{(x - 488)980}{x + 488} \text{ cm. per sec. per sec.},$$

$$\therefore \quad \tfrac{1}{2}\,\frac{x-488}{x+488}\,980 \text{ cm.} = \text{sec.}^2,$$

$$10 \text{ sec.}, \ \therefore \ \frac{100}{2} \quad \frac{x-488}{x+488}\,980 \text{ cm.}$$

Now, this is equal to 200 cm.

$$\therefore \quad \frac{x-488}{x+488}\,245 = 1,$$

from which $$x = 492.$$

Answer—492 gm.

EXERCISE XXVII.

1. A mass of 200 grammes is acted on by a force equal to the weight of 10 grammes for 20 seconds. What distance will the mass have passed through, and what velocity will it have acquired?

2. A body whose mass is 108 lbs. is placed on a smooth horizontal plane, and under the action of a certain force describes from rest a distance of $11\frac{1}{9}$ feet in 5 secs.; what is the force in British absolute units?

3. A body resting on a smooth horizontal table is acted on by a horizontal force equal to the weight of 2 ounces, and moves on the table over a distance of 10 feet in 2 seconds; find the mass of the body.

4. It is found that a body, considered as a point, has its velocity increased by 7 feet per second in any second of its motion; it is known that the body weighs 23 lbs.; what is the magnitude of the force producing the acceleration? How many pounds of matter would this force support against gravity in a place where $g = 32{\cdot}2$?

5. How long must a force of 14 pounds act on a mass of 1,000 tons to give it a velocity of one foot per second?

6. Calculate in pounds the moving force which, acting for a minute upon the mass of a ton, will get up in it the velocity of 30 miles an hour.

7. A force equal to the weight of one lb. acts on a mass of 2 lbs. for one second; if the value of g be 32, find the velocity of the mass and the space which it has travelled over. At the end of the first second the force ceases to act; how much farther will the body move during the next minute?

8. How far will a lateral pressure of an ounce move a pound on a smooth horizontal plane in five minutes?

9. Taking 10 pounds as the unit of mass, a minute as the unit of time, and a yard as the unit of length; compare the resulting systematic unit of force with that of the ft.-lb.-sec. system.

10. Determine the unit of time in order that the foot being the unit of length, the value of the intensity of gravity may be expressed by 1 instead of $32{\cdot}2$.

11. A fine string, carrying two unequal masses at its extremities, is hung over a smooth pulley. It is observed that at the end of 5 seconds the heavier weight is descending with a velocity of g feet per second. Find the ratio of the masses.

12. Two weights of 5 pounds and 7 pounds are connected by a string passing over a fixed smooth pulley; the system having no initial motion, find the velocity after three seconds.

13. Two masses of 48 and 50 grammes respectively are attached to the string of an Attwood's machine; and starting from rest, the larger mass passes through 10 centimetres in one second. Determine from these data the value of the acceleration due to gravity, the units being the centimetre and the second.

14. In Attwood's machine one of the boxes is heavier than the other by half an ounce. What must be the load of each in order that the over-weighted box may fall through one foot during the first second?

15. In an Attwood's machine—neglecting friction and the inertia of the wheels —if the two weights attached to the string be each $5\frac{1}{2}$ ounces, what must the moving weight be, in order that at the end of one second these weights may be moving with a velocity of one foot per second? In this case, how far have the weights moved from rest in the first second?

16. In Attwood's machine, where the weights are 17 oz. and 16 oz., find the acceleration and the tension of the cord.

17. The two ends of a string passing over the pulley of an Attwood's machine are loaded as follows:—A with $16\frac{1}{2}$ and B with $15\frac{1}{2}$ ounces. Find the tension at A, when it is in motion downwards.

18. The weights at the extremities of a string which passes over the pulley of an Attwood's machine are 500 and 502 grammes. The larger weight is allowed to descend; and 3 seconds after motion has begun 3 grammes are removed from the descending weight. What time will elapse before the weights are again at rest?

19. A smooth inclined plane, whose height is one half of its length, has a small pulley at the top over which a string passes. To one end of the string is attached a mass of 12 lbs., which rests on the plane; while from the other end, which hangs vertically, is suspended a mass of 8 lbs.; and the masses are left free to move. Find the acceleration and the distance traversed from rest by either mass in 5 seconds.

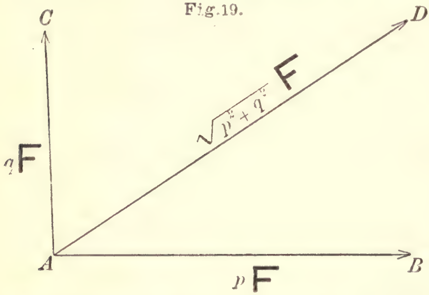
20. A mass of $6\frac{3}{4}$ lbs. is put on a smooth horizontal table and connected by a fine thread to a mass of $1\frac{1}{4}$ lbs., which hangs over the edge of the table; if the latter body is allowed to fall, dragging the former after it, what force does it exert on the former body, and what is its accelerative effect on the velocity of that body?

SECTION XXVIII.—COMPOSITION OF FORCES.

ART. 145.—**Composition of Forces.** Force involves mass-vector in its idea (Art. 141), but as the mass acted on is constant, the different forces can be represented as simple vectors. The magnitudes of the vectors must be made proportional to the magnitudes of the forces, and be drawn in corresponding directions.

Suppose that the particle at A (Fig. 19) is simultaneously subject to the force p **F** in a horizontal direction, and to the force q **F** in a vertical direction. Draw AB representing p **F** according to some scale, and AC representing q **F** according to the same scale. The diagonal AD of the rectangle formed by AB, AC represents their resultant on the same scale. Hence the magnitude of the resultant is $\sqrt{p^2+q^2}$ **F**, and its direction is the angle which has the tangent

Fig. 19.



$$q \mid p \ \mathbf{L} \text{ opp. per } \mathbf{L} \text{ adj.}$$

When the two forces are not inclined at a right angle, the construction leads to a parallelogram instead of a rectangle.

ART. 146.—**Coefficient of Friction.** Friction is a force which opposes the sliding of one body over another. Its utmost amount is proportional to the force with which the two surfaces press against each other, provided the surfaces are plane. Hence

$$\mu \ \mathbf{F} \text{ resistance} = \mathbf{F} \text{ normal pressure.}$$

The constant μ is greater when motion does not take place, than when it does take place. In the former case it is called the coefficient of *statical* friction, in the latter the coefficient of *kinetic* friction.

The weight of a body resting on a plane inclined to the horizon at α° is $mg\,\mathsf{F}$ downwards. The components of this force along and normal to the plane are $mg \sin\alpha\ \mathsf{F}$ down the plane, and $mg \cos\alpha\ \mathsf{F}$ normal. Hence the frictional resistance is $\mu mg \cos\alpha\ \mathsf{F}$ up the plane. Motion will be about to ensue when

$$mg \sin\alpha - \mu mg \cos\alpha = 0,$$

$$i.e., \qquad \mu = \tan\alpha.$$

The limiting angle at which sliding just does not commence is called the *angle of repose;* its tangent is equal to the coefficient of statical friction.

Examples.

Ex. 1. A weight of 20 lbs. rests on a horizontal plate which is made to ascend *first* with a constant velocity of one foot per second, *second* with a velocity constantly increasing at the rate of one foot per second per second; find in each case the pressure on the plate.

Take the acceleration due to gravity at 32 feet per second per second.

In the *first* case there is a force on the weight of 20×32 poundals downwards and no force on the plate, since the velocity is constant; therefore the pressure on the plate is 640 poundals.

In the *second* case there is as before a force of 20×32 poundals downwards on the weight, and there is in addition its resistance due to the acceleration imparted by the ascending plate, namely 20×1 poundals, hence the whole pressure on the plate is

$$20 \times (32 + 1) \text{ poundals, that is, } 660 \text{ poundals.}$$

Ex. 2. Forces equal to the weights of 2 lbs., 3 lbs., 4 lbs., 5 lbs. act on a particle in the directions north, east, south, west respectively; find the magnitude and direction of the resultant force.

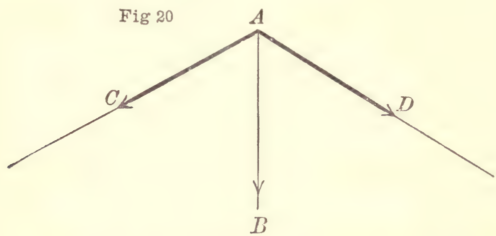
Here the intensity of gravity enters as a factor in each of the forces, hence the vectors representing the forces may be taken as $2\,\mathsf{L}$ north, $3\,\mathsf{L}$ east, $4\,\mathsf{L}$ south, $5\,\mathsf{L}$ west. The first and third compounded give $2\,\mathsf{L}$ south, and similarly the second and fourth

compounded give 2 L west; hence the resultant vector is $2\sqrt{2}$ L south-west, and the resultant force is $2\sqrt{2}$ lbs. in the direction south-west.

Ex. 3. Two rafters, making an angle of 120°, support a gasalier weighing one cwt.; what is the pressure along each rafter?

Draw the vector AB representing the force of 112 lbs. (Fig. 20). Draw BC parallel to AD and BD parallel to AC. It may be shown that ACB is an equilateral triangle; therefore the vector AC has the same length as AB; and so has AD. Hence the pressure along each rafter is 112 lb. weight.

Fig 20



Ex. 4. The slope of a plane is 45°; find the time in which a body would slide from rest down 100 feet of its length, the angle of friction being 15°.

The intensity of the weight is 32 poundals vertical per lb.,

i.e., 32 sin 45° poundals downwards per lb. + 32 cos 45° poundals normal per lb.

But tan 15° poundals up per lb. = poundal normal per lb.,

$\therefore$ 32 cos 45° tan 15° poundals up per lb.,

$\therefore$ (32 sin 45° − 32 cos 45° tan 15°) poundals down per lb.,

1 foot per second per second = poundal per lb.;

$$\therefore \quad \frac{32}{\sqrt{2}}(1 - \tan 15°) \text{ feet per second per second,}$$

$$\therefore \quad \frac{16}{\sqrt{2}}(1 - \tan 15°) \text{ feet} = \text{second}^2,$$

100 feet;

$$\therefore \qquad \frac{100\sqrt{2}}{16\,(1-\tan 15)} \text{ second}^2,$$

$$\therefore \qquad \tfrac{10}{4}\sqrt{\frac{\sqrt{2}}{1-\tan 15}} \text{ second},$$

i.e., $\qquad$ 3·5 second.

EXERCISE XXVIII.

1. A man steps on to an elevator, which thereupon descends with a uniform acceleration of 10 feet per second per second. What sensation will he experience, and calculate its amount?

2. A mass of 20 lbs. is placed upon a horizontal plane which is made to descend with a uniform acceleration of 30 feet per second per second. Find the pressure on the plane.

3. A balloon is ascending vertically with a velocity which is increasing at the rate of 3 feet per second per second; find the apparent weight of one pound weighed in the balloon by means of a spring balance.

4. Three ropes are tied together, and a man pulls at each. If, when their efforts are in equilibrium, the angle between the first and second rope is 90°, and that between the first and third is 150°, what are the relative strengths of the men as regards pulling?

5. A particle is acted on by a force whose magnitude is unknown, but whose direction makes an angle of 60° with the horizon; the horizontal component of the force is known to be 1·35 dynes. Determine the total force and also its vertical component.

6. Four forces of 24, 10, 16, 16 dynes act on a particle, the angle between the first and second being 30°, between the second and third 90°, and between the third and fourth 120°. Calculate the magnitude of their resultant.

7. Three forces, proportional to 1, 2, 3 respectively, act on a point; the angle between the first and second is 60°; the angle between the second and third is 30°. Find the angle which the resultant makes with the first.

8. On a smooth plane, rising 2 in 5, a weight of 10 lbs. is kept from sliding by a force in the direction of the plane. Determine the pressure on the plane.

9. Three cords are tied together at a point. One of these is pulled in a northerly direction with a force of 6 pounds, and another in an easterly direction with a force of 8 pounds. With what force must the third cord be pulled in order to keep the whole at rest?

10. A weight of 10 tons is hanging by a chain 20 feet long. Find how much the tension in the chain is increased by the weight being pulled out by a horizontal force to a distance of 12 feet from the vertical through the point of support.

11. A weight of 4 pounds is suspended by a string, and it is also acted on by a horizontal force. If, in the position of equilibrium, the tension of the string is 5 pounds, what is the horizontal force?

12. A mass of 10 lbs. is supported by strings of lengths 3 and 4 feet respectively, attached to two points in the ceiling 5 feet apart. What is the tension of each string?

13. Answer the above question in the case of equal strings, of such a length that the mass is only an inch from the ceiling.

14. A body rests on a horizontal plane, whose coefficient of friction is 1/2; at what inclination must a force equal to the weight of the body be applied so that it may be just on the point of moving the body?

15. Suppose the resistance of the air to the motion of a hailstone to be equal to one tenth of the weight, when the speed is 16 feet per second, and the increase to be as the square of the speed. What is the greatest speed the hailstone can acquire by falling?

16. A weight of 112 lbs. rests on a rough plane, inclined at 15° to the horizon, and the coefficient of friction is ·75. Calculate the limiting forces which, applied horizontally, must be exceeded to give the weight upward, downward, and horizontal movement respectively on the plane.

SECTION XXIX.—DEFLECTING FORCE.

ART. 147.—**Deflecting Force.** The force required to produce a given curvature of the path of a body moving with a given speed is proportional to the mass of the body and to the necessary acceleration. The body has a tendency to move uniformly in a straight line, and this tendency, looked upon as a force urging it outwards from the curve, is called *centrifugal force.* According to Art. 121 the acceleration is given by the equivalence

$$1\ \mathsf{L} \text{ per } \mathsf{T} \text{ per } \mathsf{T} = (\mathsf{L} \text{ per } \mathsf{T})^2 \text{ by (radian per } \mathsf{L} \text{ arc)},$$

hence the force is given by

$$1\ \mathsf{F} = \mathsf{M} \text{ by } (\mathsf{L} \text{ per } \mathsf{T})^2 \text{ by (radian per } \mathsf{L} \text{ arc)}.$$

Now, the units upon which F per M depends can be transformed into any form in which they are combined into units, provided the quality of each unit is preserved; thus

$$\begin{aligned} 1\ \mathsf{F} &= \mathsf{M} \text{ by } (\mathsf{L} \text{ arc per } \mathsf{T})^2 \text{ per } \mathsf{L} \text{ radius}, \\ &= \mathsf{M} \text{ by } (\mathsf{L} \text{ arc per } \mathsf{T}) \text{ by (radian per } \mathsf{T}), \\ &= \mathsf{M} \text{ by (radian per } \mathsf{T})^2 \text{ by } \mathsf{L} \text{ radius}. \end{aligned}$$

As 2π radian = revolution, by substituting in the last, we get

$$4\pi^2 \mathsf{F} = \mathsf{M} \text{ by (revolution per } \mathsf{T})^2 \text{ by } \mathsf{L} \text{ radius.}$$

In the case of the British system we have

$$1 \text{ poundal} = \text{lb. by (ft. arc per sec.)}^2 \text{ per ft. radius.}$$

In the C.G.S. system

$$1 \text{ dyne} = \text{gm. by (cm. arc per sec.)}^2 \text{ per cm. radius.}$$

ART. 148.—**Simple Harmonic Motion.** The acceleration on a particle describing a simple harmonic motion is given by the equivalence

$$1 \mathsf{L} \text{ per } \mathsf{T} \text{ per } \mathsf{T} = (\text{radian per } \mathsf{T})^2 \text{ by } \mathsf{L} \text{ displacement,}$$

therefore the force required is given by

$$1 \mathsf{F} \text{ per } \mathsf{M} = (\text{radian per } \mathsf{T})^2 \text{ by } \mathsf{L} \text{ displacement.}$$

In the motion of a simple pendulum, instead of L displacement we have L length of pendulum; hence

$$1 \mathsf{F} \text{ per } \mathsf{M} = (\text{radian per } \mathsf{T})^2 \text{ by } \mathsf{L} \text{ length,}$$

or $$4\pi^2 \mathsf{F} \text{ per } \mathsf{M} = (\text{revolution per } \mathsf{T})^2 \text{ by } \mathsf{L} \text{ length.}$$

EXAMPLES.

Ex. 1. If a stone weighing 2 lbs. be attached to a yard of string, which can just support 20 lbs., at what rate must it be whirled round horizontally so as to break the string?

$$4\pi^2 \text{ poundal} = \text{lb. by (revolution per sec.)}^2 \text{ by ft. radius,}$$

2 lbs. by 3 ft. radius,

$$\therefore \quad 24\pi^2 \text{ poundal} = (\text{revolution per sec.})^2.$$

But $$20 \times 32 \text{ poundals,}$$

$$\therefore \quad \frac{20 \times 32}{24\pi^2} (\text{revolution per sec.})^2,$$

$$i.e., \quad \frac{15 \times 4^2}{3^2 \times \pi^2} (\text{revolution per sec.})^2,$$

$$i.e., \quad \frac{\sqrt{15} \times 4}{3\pi} \text{ revolution per second,}$$

$$i.e., \quad 1{\cdot}6 + \text{ revolution per second.}$$

Ex. 2. A toy-car, whose mass is $\frac{1}{2}$ lb., runs at the rate of 5 miles an hour on a level circular railway 20 feet in circumference; calculate the horizontal pressure on the rails.

1 poundal = lb. by (ft. per sec.)^2 per ft. radius,

$\frac{1}{2}$ lb. by $\left(\frac{22}{3}\right)^2$ (ft. per sec.)^2 per $\frac{20}{2\pi}$ ft. radius.

$$\therefore \qquad \frac{22^2 \times \pi}{2 \times 3^2 \times 10} \text{ poundals,}$$

$$i.e., \qquad \frac{22 \times 22 \times 22}{2 \times 9 \times 10 \times 7} \text{ poundals,}$$

$$i.e., \qquad 8{\cdot}4 \text{ poundals.}$$

Ex. 3. A railway train moves smoothly at the rate of 30 miles an hour over a curve of 500 yards radius; find the angle at which a plumb-line in one of the carriages will be inclined to the vertical.

1 poundal per lb. = (ft. per sec.)^2 per ft. radius,

$\frac{(44)^2}{500 \times 3}$ (ft. per sec.)^2 per ft. rad.,

$$\therefore \qquad \frac{(44)^2}{500 \times 3} \text{ poundals per lb.,}$$

$$\therefore \qquad \frac{(44)^2}{500 \times 3} \text{ ft. outwards per sec. per. sec.}$$

Also 32·2 ft. downwards per sec. per sec.;

$$\therefore \qquad \frac{(44)^2}{500 \times 3 \times 32{\cdot}2} \text{ ft. outwards per ft. downwards,}$$

$$i.e., \qquad 0{\cdot}4 \text{ ft. outwards per ft. downwards.}$$

Hence the tangent of the angle is ·04, and the angle about 6°.

Ex. 4. Find the maximum force in a case of simple harmonic motion in which the moving mass is a gramme, the range on each side of the middle position 10 centimetres, and the period $\frac{1}{300}$ second.

$4\pi^2$ dyne = gm. by (rev. per sec.)^2 by cm. displacement,

1 gm. by 300^2 (rev. per sec.)^2 by 10 cm.,

$\therefore$ $4\pi^2 \times 300^2 \times 10$ dynes,

i.e., $36\pi^2 \times 10^5$ dynes,

i.e., $3{\cdot}6\pi^2$ megadynes.

Ex. 5. Find how many vibrations a simple pendulum, 4 inches long, would make in a minute at Glasgow.

$4\pi^2$ poundals per pound = (rev. per sec.)^2 by ft. length,

$\frac{1}{3}$ ft. length,

$\therefore$ $\frac{4\pi^2}{3}$ poundals per pound = (rev. per sec.)^2,

32·2 poundals per pound at Glasgow;

$\therefore$ $\frac{32{\cdot}2 \times 3}{4\pi^2}$ (revolutions per sec.)^2,

$\therefore$ $\frac{\sqrt{96{\cdot}6}}{2\pi}$ revolutions per sec.,

60 sec. $\therefore$ $\frac{30\sqrt{96{\cdot}6}}{\pi}$ revolutions,

i.e., 94 complete vibrations,

or 188 single vibrations.

Ex. 6. The time of a complete vibration at Paris of a pendulum 6,400 centimetres long is 16 seconds; show that the value of g is 986 approximately.

$4\pi^2$ dyne per gm. = (rev. per sec.)^2 by cm. length,

$\frac{1}{16^2}$ (rev. per sec.)^2 by 6,400 cm. length,

$\therefore$ $\frac{4\pi^2 \times 6400}{16^2}$ dyne per gm.,

i.e., $\pi^2 \times 100$ dyne per gm.,

i.e., 986·96 dyne per gm.,

$\therefore$ 986 dyne per gramme approximately.

EXERCISE XXIX.

1. Calculate, in pound's weight, the tension of a string 4 feet long which has a mass of 10 pounds attached to it, describing a horizontal circle once in half a second.

2. Two bodies, A and B, describe circles with constant velocities; the radius of A's circle is 390 times that of B's; A moves round its circle once while B moves round its circle 13 times; A's mass is 91 times B's mass. Compare the force acting on A with the force acting on B.

3. A mass of 20 lbs. is revolving uniformly, once in 5 seconds, in a circle whose radius is 3 feet. Find the centrifugal force.

4. A mass of 1 lb. is placed on the rim of a wheel 2 ft. in diameter, which revolves upon its axis, and is otherwise balanced. The linear velocity of the rim being 30 ft. per sec., what is the pull on the axis as caused by the mass of 1 lb.

5. A locomotive of 20 tons runs at the rate of 30 miles an hour at a part of the line where the radius of curvature is 10 miles; calculate in tons its entire pressure against the inner surfaces of the rails.

6. A locomotive, 15 tons in weight, runs with a velocity of 20 miles an hour in a circle of a mile radius; calculate in poundals its entire pressure against the rails.

7. A railway carriage weighing 4 tons is passing round a curve, the radius of which is 250 yards, at the rate of 20 miles an hour; what is the outward pressure on the rails?

8. If the earth were set to revolve on its axis in half a day, nothing else being altered, what would be the weight of a mass of 100 lbs., tested by a spring balance, supposing the balance to have been graduated to show pounds' weight at the equator, when the earth was revolving with its actual angular velocity? Earth's radius is 21×10^6 feet; $g = 32{\cdot}09$ at the equator, with actual angular velocity.

9. A railway carriage is going round a curve of 500 feet radius, at 30 miles per hour. Find how much a plummet hung from the roof by a thread 6 feet long would be deflected from the vertical.

10. A skater describes a circle of 100 feet radius, with a velocity of 18 feet per second; what is his inclination to the ice?

11. Find the force per gramme of the earth's mass towards the sun, supposing the earth's motion to be in a circle of radius $1{\cdot}473 \times 10^{13}$ centimetres.

12. Calculate the tension of an endless chain, of 1 pound per foot, and 30 feet long, when made to rotate as a horizontal circle once per second.

13. Masses of 6 and 16 M are fixed at the ends of a horizontal rod; round what vertical axis must it revolve that the two centrifugal forces may destroy one another; *first* when the rod is supposed without mass, *second* when its mass is 4 M.

14. What ought to be the difference of level between the rails, when the radius of curvature of a railway curve is 300 yards, the breadth of the guage 4 ft. $8\frac{1}{2}$ in., and the highest velocity of a train at the place 45 miles an hour?

15. A square frame-work of a yard in the side rotates once per second about

one of its sides, which is vertical; what must be the coefficient of friction to keep a ring from slipping down the opposite side?

16. A weight of 28 lbs. is suspended by means of a ring on a straight rod which revolves in a horizontal plane about a fixed axis; the coefficient of friction between the ring and the rod is $\frac{1}{5}$, and the distance of the weight from the axis is 5 ft.: find the angular velocity when the weight will begin to move outwards.

17. A mass of 2 pounds is kept performing simple harmonic vibrations, at the rate of 30 periods per second, through a range of a tenth of a foot on each side. Find the maximum force, and the force at 1/50 foot from the middle position.

18. A mass of 1/1000 pound vibrates 256 times in a second through a range of 1/10 inch. Find the maximum force upon it.

19. A mass of a gramme vibrates through a millimetre on each side of its middle position 256 times per second; find the maximum force upon it in grammes weight. Assume the intensity of gravity at 981·4 centimetres per second per second.

20. A body whose mass is 10 lbs. is tied to a thread 6 ft. long, and is allowed to swing backwards and forwards through the arc of a semi-circle; when it is 30° from the lowest point of the arc, what forces are acting on it?

21. A seconds pendulum is lengthened and the time of oscillation is thereby increased by an eighth of a second. Calculate the increase in length. ($g = 32·2$.)

22. Given that the intensity of gravity at Paris is 9·81 metres per second; what is the length of the pendulum which beats seconds there.

23. A pendulum, 39·20 inches long, vibrates seconds of mean time at a certain place. Find the force of gravity per pound of matter at that place. How much faster would the pendulum beat at a place where the force of gravity is a quarter per cent. greater?

24. What would be the length of a simple pendulum vibrating in 2·5 seconds, at a place where the intensity of gravity is 31·5 ft. per sec. per sec.

25. What must be the length of a pendulum beating 10 times in a minute if the seconds pendulum is 39 inches long.

26. A pendulum 37·8 inches long is found to make 182 beats in 3 minutes at a certain place; find the force of gravity at the place.

27. At a place where a simple pendulum 100 centimetres long beats seconds, find how many seconds are gained per day when the pendulum is shortened by one millimetre.

28. A clock, whose pendulum ought to beat seconds, is gaining at the rate of $\frac{1}{2}$ hour per week. How many turns should be given to the screw-head, supposing it to have 40 turns to the inch, to correct the error in length of the pendulum. Take the seconds pendulum at $39\frac{1}{8}$ inches.

SECTION XXX.—SPECIFIC GRAVITY.

ART. 149.—**Heaviness and Density.**—We have

$$1\ \mathsf{F} = \mathsf{M} \text{ by } \mathsf{L} \text{ per } \mathsf{T} \text{ per } \mathsf{T}.$$

The acceleration at a particular place due to the attraction of the earth is the same for every kind of matter; let it be denoted by g L per T per T, then

$$g\ \mathsf{F} = \mathsf{M}.$$

Let the density of a substance be ρ $\mathsf{M} = \mathsf{V}$, by eliminating M we deduce

$$\rho g\ \mathsf{F} = \mathsf{V}.$$

This gives us the idea of *weight* per *volume*, that is, of *heaviness.**

ART. 150.—**Specific Gravity.** By the *specific gravity* of a substance is meant its heaviness compared with the heaviness of a standard substance, as water at the temperature of 62° Fahr. (British), or at its temperature of maximum density (French). As the intensity of gravity is the same for all kinds of matter, the specific gravity, that is, the relative heaviness, has the same value as the specific mass, that is, the relative density.

ART. 151.—**Buoyancy.** Let the heaviness of a solid be

$$\rho g\ \mathsf{F} = \mathsf{V},$$

and of a fluid

$$\rho' g\ \mathsf{F} = \mathsf{V}.$$

When the solid is immersed in the fluid, the heaviness of the fluid acts as an upward force, and the resulting heaviness of the solid is

$$\rho g - \rho' g\ \mathsf{F} = \mathsf{V},$$

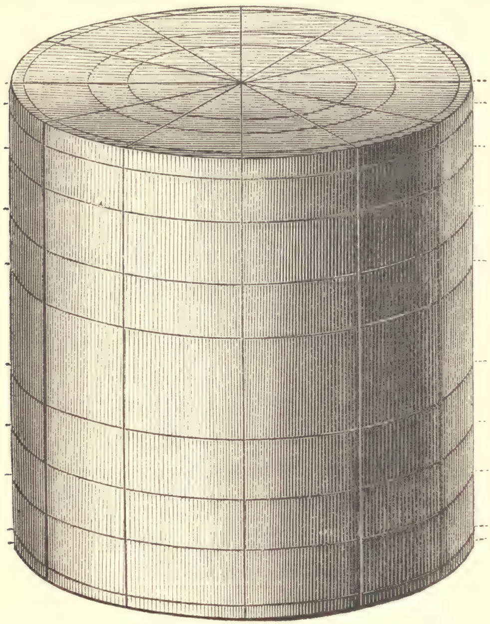
$$i.e.\quad (\rho - \rho')g\ \mathsf{F} = \mathsf{V}.$$

When ρ' is greater than ρ, the resulting heaviness becomes negative, that is, it becomes *buoyancy*.

The determination of the density of water at 4° C. upon which the kilogramme was based, was effected by weighing a brass

* Rankine's *Rules and Tables*, p. 102

cylinder in water. The cylinder used is represented in the accompanying illustration. It was hollow but completely closed,



except that a small tube kept up the connection between the air in the cylinder and the air in the room when the cylinder was immersed in water. The ratio of hollow to solid cylinder was so arranged that the weight of the metal was only a little greater than the buoyancy of the water displaced. The diameter and height were each intended to be 2·435 decimetres. By means of the lines drawn on its surface the mean diameter was found at 17° C. to be 2·428368 dm., and the mean height 2·437672 dm. The volume of the cylinder at 0° C. was computed to be 11·28 cubic decimetres, and that of the solid part 1·506 cb. dm. The weighing was effected by means of a provisional kilogramme and

its decimal parts. Let the provisional kilogramme be denoted by kilo; then the final results of the weighings at 60° C. were as follows:—

Weight of the cylinder in air, 11·4660055 kilo,
weight of the cylinder in distilled water, 0·1967668 kilo;
therefore weight of water having the volume
of the cylinder, 11·2692387 kilo;

therefore 11·2692387 kilo = 11·28 cubic decimetres,

i.e. ·999046 kilo = cubic decimetre.

Corrected for the change to 4° C., it is

·999207 kilo = cubic decimetre,

and the primary kilogramme was constructed to equal

·999207 kilo.

Examples.

Ex. 1. How much would a brass kilogramme appear to weigh if it were suspended in water?

$$8{\cdot}3g \text{ dynes} = \text{cc. of brass},$$
$$g \text{ dynes} = \text{cc. of water},$$
$$\therefore \quad (8{\cdot}3 - 1)\, g \text{ dynes apparent} = 8{\cdot}3g \text{ dynes real},$$
$$1{,}000\, g \text{ dynes real},$$
$$\therefore \quad \frac{1{,}000\,(8{\cdot}3 - 1)}{8{\cdot}3}\, g \text{ dynes apparent},$$
$$\textit{i.e.,} \quad 879{\cdot}5\, g \text{ dynes apparent}.$$

Observation.—g dynes = gm., hence gm. may be substituted in the above instead of g dynes.

Ex. 2. A solid weighs 100 grains in water and 120 grains in alcohol of specific gravity 0·8. Find the mass and the specific gravity of the solid.

Suppose that the mass of the solid is x grains. Then $(x - 100)$ grains is its apparent loss of mass in water, $\therefore$ $(x - 100)$ grains is

the mass of water displaced. Similarly $(x-120)$ grains is the mass of the alcohol displaced; hence

$$\frac{x-120}{x-100} \text{ grain alcohol} = \text{grain water}.$$

But $\qquad\qquad \cdot 8$ grain alcohol = grain water.

$$\therefore \quad \frac{x-120}{x-100} = \cdot 8,$$

from which $\qquad x = 200.$

Hence, $\quad$ 200 grains solid = 200 − 100 grains water;

$\therefore \quad$ 2 grains solid = grain water.

Ex. 3. A piece of copper wire 4 metres long weighs 17·2 grammes in air, and 15·2 grammes in water; find the diameter of the wire.

As the heaviness of air is small, the weight of the wire in air may be taken as its true weight.

Mass of the water displaced is 17·2 − 15·2 grammes, *i.e.*, 2 gms.

1 gm. of water = sq. cm. by cm. length,

$\therefore \quad$ 2 sq. cm. by cm. length;

400 cm. length, $\quad \therefore \quad \frac{2}{400}$ sq. cm.

$$\frac{\pi}{4} \text{ sq. cm.} = (\text{cm. diam.})^2;$$

$$\therefore \quad \frac{2}{100 \times \pi} (\text{cm. diam.})^2,$$

$$\therefore \quad \sqrt{\frac{2}{100\pi}} \text{ cm. diam.}$$

$$\begin{aligned} \log 2 &= \cdot 30103 \\ \log 100\pi &= 2 \cdot 49715 \\ \hline 2)\bar{3}&\cdot 80388 \\ \hline \bar{2}&\cdot 90194 \end{aligned}$$

Hence $\qquad$ ·0798 cm. diameter.

Ex. 4. Find what weight of lead attached to 20 lbs. of cork would be just sufficient to sink it in water.

·24 lb. cork = lb. water,

$\therefore$ $\cdot 24$ poundals, weight of cork = poundal, weight of water,

$\therefore$ $\dfrac{\cdot 24 - 1}{\cdot 24}$ poundals app. wt. of cork = poundal, real wt. of cork,

$20g$ poundals, real weight of cork,

$\therefore$ $20g \, \dfrac{\cdot 24 - 1}{\cdot 24}$ poundals app. weight of cork.

Similarly

$\dfrac{11 \cdot 4 - 1}{11 \cdot 4}$ poundals app. wt. of lead = poundal real wt. of lead.

But the apparent weight of the lead is to be equal to the apparent weight of the cork, only in the opposite direction; hence

$$20g \, \frac{\cdot 24 - 1}{\cdot 24} \, \frac{11 \cdot 4}{11 \cdot 4 - 1} \text{ poundals real weight of lead}$$

$$\therefore \quad 20 \, \frac{\cdot 24 - 1}{\cdot 24} \, \frac{11 \cdot 4}{11 \cdot 4 - 1} \text{ pounds of lead,}$$

i.e., 69·4 pounds of lead.

Ex. 5. A rectangular barge open at top is made of sheet iron a quarter of an inch thick (480 lbs. per cubic foot); it is 36 feet long, 12 feet wide and 7 feet deep: what weight placed in the barge will just sink it?

For the rectangular sides

1 square foot = foot long by foot deep,

2 (36 + 12) feet long by 7 feet deep,

$\therefore$ 2 (36 + 12) 7 square feet.

For the rectangular bottom

36×12 square feet,

hence surface of plate $2\,(36 + 12)\,7 + 36 \times 12$ square feet,

and 480 lbs. = square foot surface by foot thick,

and $\dfrac{1}{4 \times 12}$ foot thick,

$$\therefore \quad 480 \, \frac{14 \times 48 + 9 \times 48}{48} \text{ lbs.,}$$

i.e., 480×23 lbs. *i.e.*, 11,040 lbs.

Since the barge is rectangular

1 cubic foot = foot long by foot broad by foot deep,

36 feet by 12 feet by 7 feet,

$\therefore$ $36 \times 12 \times 7$ cubic feet,

62·5 lbs. = cubic foot of water,

$\therefore$ $62{\cdot}5 \times 36 \times 12 \times 7$ lbs.,

i.e., 189,000 lbs.

Hence weight required is 189,000 lbs. − 11,040 lbs.,

i.e., 178,000 lbs.

Ex. 6. What depth of water is required to float an iceberg one mile square by 500 feet high?

·92 **M** of ice = **M** of pure water,

$\therefore$ 1 foot^3 of ice = ·92 foot^3 of pure water;

but the cross-section is constant,

$\therefore$ 1 foot height of ice = ·92 foot height of pure water.

Similarly 1·025 feet of pure water = foot of sea water,

500 feet of ice,

$\therefore$ $\dfrac{500 \times {\cdot}92}{1{\cdot}025}$ feet of sea water,

i.e., 449 − „

EXERCISE XXX.

1. A piece of copper weighs 31 grains in air and $27\frac{1}{2}$ in water. Find its specific gravity.

2. A solid, of which the volume is 1·6 cubic centimetres weighs 3·4 grammes in a fluid of specific gravity 0·85. Find the specific gravity and weight of the substance.

3. A glass ball weighs 3,000 grains, and has a specific gravity 3. What will be its apparent weight when immersed in a liquid whose specific gravity is 0·92?

4. A piece of brass, whose specific gravity is 8·4 and weight 20 grammes, is attached to one end of a string which passes over a smooth pulley, and hangs in water. What weight must be attached to the other end of the string that there may be equilibrium?

5. If a piece of wood exactly counterpoises a cubic inch of brass, what is its volume, supposing the specific gravity of the wood to be 0·9, that of the brass 8·1, and that 800 cubic inches of the air in which the weighing takes place, are as heavy as a cubic inch of water at the standard temperature?

6. A body weighs 8 oz. in air, 6·17 oz. in olive oil, and 5·948 oz. in sea water. Compare the specific gravities of olive oil and sea water.

7. A body weighs in air 80 grains, in water 56 grains, and in another liquid 46 grains. Find the specific gravity of this liquid.

8. A piece of oak having a specific gravity of ·74 and a volume of 32 cubic inches floats in water. How much water will it displace?

9. If a piece of wood, weighing 120 pounds, floats in water with $\frac{4}{5}$ of its volume immersed, show what is its whole volume.

10. An inch cube of a substance of specific gravity 1·2 is immersed in a vessel containing two fluids which do not mix; the specific gravities of these fluids are 1·0 and 1·5. Find what will be the point at which the solid will rest.

11. An inch cube of ice having a specific gravity ·918 is floating in water of which the specific gravity is 0·99987. Find its height above the surface of the water.

12. A body floats in water with $\frac{1}{8}$ of its volume above the surface; determine its specific gravity. How much of it will be submerged in a fluid whose specific gravity is ·9?

13. Suppose that an iceberg has the form of a cube, and floats flat with a height of 30 feet above the ocean. What depth will it have under the surface?

14. The specific gravity of lead is 11·4, that of cork 0·24. How much lead must be attached to a piece of cork weighing 3 grammes to make it sink?

15. Find whether a piece of cork weighing 2 grammes with a piece of lead weighing 6·94 grammes attached to it, will sink or swim in water.

16. A body of 300 grammes having a specific gravity 5, has 100 grammes of another substance attached to it, and the joint weight of the two in water is 300 grammes. Find the specific gravity of the attached substance.

17. A certain body A is observed to float in water with exactly half its volume submerged; and, when attached to another B of twice its own volume, it is found to be just submerged. Find the specific gravity of A and of B.

18. Find the specific gravity of a piece of cork from the following data—

Weight of cork in air, - - - 2 grammes,
Weight of cork and sinker in water, 4 grammes,
Weight of sinker in water, - - 12 grammes.

19. A cube of metal is floating in mercury sp. gr. 13·6; when a weight of 170 lbs. is placed on the top, the cube is observed to sink 3 inches. Find the size of the cube.

20. A body composed partly of iron and partly of a wood (sp. gr. 0·4) is observed to be just submerged in water. Compare the volumes of iron and of wood in the body.

21. An iron ball of 12 lbs. floats in mercury covered with water. Find the weights of the parts in the two fluids.

22. A rod of uniform section is formed partly of platinum and partly of iron. The platinum portion being 2 inches long, what will be the length of the iron portion, when the whole floats in mercury with one inch above the surface?

23. How much iron must be attached to a wooden beam 10 feet long, and 5 inches broad and thick, in order to sink it? The specific gravity of the wood is ·7.

24. A raft, whose weight and specific gravity are known, floats in water. Show how to determine the greatest weight which it can support without sinking.

25. An empty balloon with its car and appendages weighs in air 1,200 lbs. If a cubic foot of air weigh $1\frac{1}{4}$ oz., how many cubic feet of gas of specific gravity ·52 must be introduced before the balloon will begin to ascend?

26. A small vessel quite filled with distilled water weighs 530 grains; and when 26 grains of sand are thrown in, the whole weighs 546 grains. Find the specific gravity of the sand.

27. A piece of copper and a piece of silver fastened to the two ends of a string passing over a pulley, hang in equilibrium when entirely immersed in a liquid whose specific gravity is 1·15. Determine the relative volumes of the masses, the specific gravities of silver and copper being 10·47 and 8·89.

SECTION XXXI.—PRESSURE.

ART. 152.—**Intensity of Pressure.** When force acts across a surface, there are two equal and oppositely directed forces. When their directions are inwards towards one another, each is called a *pressure;* and when outwards from one another, each is called a *tension.* Either of these may be specified in terms of unit surface of application; hence the idea

$$\mathsf{F} \text{ per } \mathsf{L}^2.$$

It is properly called *intensity of pressure*, but owing to the frequency with which it occurs it is commonly called "pressure." In the British system we have poundals per square foot, lb. by weight per square foot, lb. by weight per square inch, etc. By "lb. by weight" is meant 32·187 poundals (Art. 144).

In the C.G.S. system we have dyne per sq. cm.,
gm. by weight per sq. cm., etc.

ART. 153.—**Height of Column of a Liquid.** When a mass of liquid is at a uniform temperature, the density is very nearly uniform throughout, though the lower layers have to support the weight of

the upper. No appreciable error is made by calculating the pressure at a point in the liquid, on the assumption that the density is uniform throughout.

Let the density of the liquid be

$$\rho\ \mathsf{M} = \mathsf{L}^3;$$

for a vertical column

$$1\ \mathsf{L}^3 = \mathsf{L}^2 \text{ horizontal surface by } \mathsf{L} \text{ height};$$

$$\therefore \quad \rho\ \mathsf{M} = \mathsf{L}^2 \text{ horizontal surface by } \mathsf{L} \text{ height}.$$

Now $\quad g\ \mathsf{F} = \mathsf{M},$

$$\therefore \quad \rho g\ \mathsf{F} = \mathsf{L}^2 \text{ horizontal surface by } \mathsf{L} \text{ height},$$

$$\therefore \quad \rho g\ \mathsf{F} \text{ per } \mathsf{L}^2 \text{ horizontal surface} = \mathsf{L} \text{ height}.$$

But the pressure is the same in magnitude, whatever be the inclination of the plane of application taken at the point, provided it is sufficiently small; hence

$$\rho g\ \mathsf{F} \text{ per } \mathsf{L}^2 \text{ surface} = \mathsf{L} \text{ height}.$$

Thus for every unit of height of a column of liquid we have ρg times the corresponding unit of pressure. This explains why a pressure is usually expressed as a height; for example, as so many inches of mercury. The equivalence in this case is

·491163 pound weight per sq. in. = in. height of mercury at 32° F.

Examples.

Ex. 1. Find the relation between pound per sq. inch and kilogramme per sq. centimetre.

$$\cdot 4536 \text{ kgm.} = \text{lb.} \qquad (1)$$

$$1 \text{ cm.} = \cdot 3937 \text{ inch},$$

$$\therefore \quad 1 \text{ cm}^2. = (\cdot 3937)^2 \text{ inch}^2. \qquad (2)$$

Dividing (1) by (2)

$$\cdot 4536 \text{ kgm. per sq. cm.} = \frac{1}{(\cdot 3937)^2} \text{lb. per sq. inch},$$

$$\therefore \quad \cdot 4536 \times (\cdot 3937)^2 \text{ kgm. per sq. cm.} = \text{lb. per sq. inch},$$

i.e., $\quad \cdot 0703$ kgm. per sq. cm. = lb. per sq. inch.

Ex. 2. Find the value of the standard atmospheric pressure in terms of dynes per sq. cm.

The density of mercury is 13·596 gm. = cc., and the standard intensity of gravity is 981 dyne per gm. Hence

$$13{\cdot}596 \times 981 \text{ dyne per sq. cm.} = \text{cm. height of column,}$$

$$\text{the standard height is } 76 \text{ cm.,}$$

$$\therefore \quad 76 \times 13{\cdot}596 \times 981 \text{ dyne per sq. cm.}$$

$$\begin{array}{r} 981 \\ 76 \\ \hline 5886 \\ 6867 \\ \hline 74556 \end{array} \qquad\qquad \begin{array}{r} 74556 \\ 69531 \\ \hline 74556 \\ 22366 \\ 3727 \\ 670 \\ 44 \\ \hline 101353 \end{array}$$

Answer—$1{\cdot}014 \times 10^6$ dyne per sq. cm.

Ex. 3. At the bottom of a mine a mercurial barometer stands at 77·4 cm.; what would be the height of an oil barometer at the same place, the specific gravity of mercury being 13·596, and that of oil 0·9 ?

$$13{\cdot}596 \text{ gm. per sq. cm.} = \text{cm. of mercury,}$$

$$\cdot 9 \quad \text{gm. per sq. cm.} = \text{cm. of oil,}$$

$$\therefore \quad 13{\cdot}596 \text{ cm. of oil} = \cdot 9 \text{ cm. of mercury,}$$

$$77{\cdot}4 \text{ cm. of mercury,}$$

$$\therefore \quad \frac{77{\cdot}4 \times 13{\cdot}596}{\cdot 9} \text{cm. of oil,}$$

$$\textit{i.e.,} \quad 1169{\cdot}3 \text{ cm. of oil.}$$

Ex. 4. The area of the plunger of a force pump being 3 square inches, find the pressure upon it when water is forced up a height of 20 feet.

$$62{\cdot}5 \text{ lb. by weight per sq. ft.} = \text{ft. height,}$$

$$20 \text{ ft. height,}$$

$$\therefore \quad 20 \times 62{\cdot}5 \text{ lb. weight} = \text{sq. ft.}$$

$$1 \text{ sq. ft.} = 144 \text{ sq. inches,}$$

$$3 \text{ sq. inches,}$$

$$\therefore \quad \frac{3 \times 20 \times 62{\cdot}5}{144} \text{lb. by weight,}$$

$$\textit{i.e.,} \qquad 26 \text{ lb. weight.}$$

Ex. 5. Find the whole pressure on a vertical lock-gate 14 feet broad, against which the water rises $9\frac{1}{2}$ feet.

The intensity of the pressure is given by

$$62{\cdot}5 \text{ lb. weight per sq. ft.} = \text{ft. height};$$

hence at the bottom it is

$$9{\cdot}5 \times 62{\cdot}5 \text{ lb. weight per sq. ft.,}$$

and at the top,

$$0 \text{ lb. weight per sq. ft.}$$

As the pressure increases uniformly with the depth, the average pressure is

$$\frac{9{\cdot}5 \times 62{\cdot}5}{2} \text{lb. weight per sq. ft.}$$

Now $\qquad 14 \times 9{\cdot}5$ sq. ft. of surface,

$$\therefore \qquad 7 \times (9{\cdot}5)^2 \times 62{\cdot}5 \text{ lb. weight,}$$

$$\textit{i.e.,} \qquad 39484 \text{ lb. weight.}$$

Ex. 6. A square board, whose edge is 2 feet long, is immersed in water with one edge just at the surface, and it is inclined to the horizontal at 30°. Find the pressure on one side of it.

$$62{\cdot}5 \text{ lb weight} = \text{ft. long by ft. broad by ft. deep,}$$

$$2 \text{ ft. long by } 2 \cos 30^\circ \text{ ft. broad by } \frac{2 \sin 30^\circ + 0}{2} \text{ft. deep,}$$

$$\therefore \quad 62{\cdot}5 \times 2 \times 2\frac{\sqrt{3}}{2} \times \tfrac{1}{2} \text{ lb. weight,}$$

$$\textit{i.e.,} \qquad 62{\cdot}5\sqrt{3} \text{ lb. weight.}$$

EXERCISE XXXI.

1. Reduce pound per square foot and pound per square inch to dyne per square cm., when the intensity of gravity is 981 cm. per sec. per sec.

2. Given the average weight of a man as 12 stones, and the density of a dense crowd as 5 men = 7 square feet. Find the pressure per square foot due to a dense crowd on a bridge.

3. Wind, having a velocity of 50 miles an hour, exerts a pressure of 12 lbs. per

square foot. Find the total force exerted on a surface 5 ft. by 6 ft. inclined at an angle of 25° to the direction of the wind.

4. Reduce 25·6, 26·7, and 27·8 inches of mercury to millimetres of mercury.

5. Express a pressure of 100 mm. of mercury in terms of inch of water.

6. Reduce a pressure of 600 millimetres of mercury at 0°C. to dynes per square cm. when the intensity of gravity is 981 cm. per sec. per sec.

7. Express a pressure of 760 mm. of mercury in terms of kilogramme per square metre.

8. Given the density of water as 62·5 lb. per cubic foot, and that 1·025 lb. of sea-water is equivalent in volume to one lb. of pure water. Find the pressure at a depth of 200 feet under the surface of the ocean due to the superincumbent water.

9. If the plunger of a force-pump has a cross-section of 8 square inches, and works 50 feet below the cistern, what pressure is required to force it down?

10. During a storm the barometer at sea-level stood as low as 27·466 inches. What was the pressure in lbs. per square inch?

11. What ought to be the length of a water barometer, inclined to the horizon at an angle of 30°, the mercury barometer standing at 30·5 inches?

12. The diameter of the tube of the barometer is 1 cm., and that of the cistern 4·5 cm. If the mercury in the tube rise through 2·5 cm., find the real alteration in the height of the barometer.

13. What is the theoretical height to which water can be raised by the common pump, when the mercurial barometer stands at 28 inches?

14. A barometer is observed to fall one tenth of an inch when carried up 88 feet of vertical height; how much would it fall if taken 132 yards up a hill rising 1 in 3?

15. A piece of metal of sp. gr. 8, and weighing 20 lbs. is dropped into a cylinder filled with water; find the additional pressure on the base.

16. What depth of water is required to float an iceberg one mile square by 500 feet high?

17. The neck of a wine bottle with flat bottom is 4 inches long, the total height of the bottle being 12 inches. When the bottle is filled with wine of specific gravity 0·99 to within half an inch of its mouth, what is the pressure on each square inch of the bottom?

18. What is the pressure on a sluice-gate 12 feet broad, against which the water rises 5 feet?

19. A sluice-gate is 4 feet broad and 6 feet deep, and the water rises to a height of 5 feet on one side and 2 feet on the other side. Find the pressure in pounds on the gate.

20. Find the whole pressure upon a vertical dam of a column of water 10 feet deep and 30 feet wide. What would be the pressure of the same head of water against a dam inclined at an angle of 45° to the horizon?

21. A vessel, consisting of a decimetre cube, is filled to one third of its height

with mercury, while the rest is filled with water; determine the whole pressure against one of the sides in kilogrammes.

22. A rectangular board, one foot square, is immersed in water with its upper edge 10 feet below the surface of the water, and horizontal, the surface of the board being vertical. Find the total pressure on one side.

23. If the height of the water barometer be 1,033 centimetres, what will be the pressure on a circular disc whose radius is 7 cm. when sunk in water to a depth of 50 metres?

24. A square plate whose area is 64 square inches is immersed in sea water, its upper edge being horizontal and 12 inches below the surface. Determine the whole pressure of the water on the plate when it is inclined at 45° to the horizon, assuming a cubic inch of sea water to weigh 0·63 ounces.

SECTION XXXII.—PRESSURE OF A GAS.

ART. 154.—**Height of Homogeneous Atmosphere.** In the case of a vertical column of gas, the density is not uniform throughout. The gas in a horizontal layer is compressed by the weight of the superincumbent gas. It is sometimes convenient to consider what would be the height of a vertical column of gas having the density throughout which it actually has at the bottom, and producing the same pressure at the bottom. The height of such a column is called *height of homogeneous atmosphere*, because the conception applies to the air of the atmosphere. Prof. Everett suggests the shorter and more appropriate name, "pressure height."*

The pressure of the atmosphere is used as a convenient unit of pressure in the same way as the weight of a pound is used as a convenient unit of force. The exact unit is defined by the following equivalences—

1 atmosphere = 29·922 inches of merc. at 32° F. (British),
= 760 mm. of merc. at 0° C. (French).

ART. 155.—**Dependence of Density on Pressure.** The law discovered by Boyle states that the density of a portion of gas is

* *Units and Physical Constants*, p. 37.

proportional to the pressure of the portion, provided that the temperature be kept constant. Hence,

$$k\ \mathsf{M}\ \text{per}\ \mathsf{V} = \mathsf{F}\ \text{per}\ \mathsf{S},$$

or

$$k\ \mathsf{M} = \mathsf{F}\ \text{per}\ \mathsf{S}\ \text{by}\ \mathsf{V}.$$

The letter k is used to denote some constant number.

Examples.

Ex. 1. A litre of air at 0° C. and 760 mm. pressure contains 1·293 gms. Find the mass of 73 litres at the same temperature, and under 1,000 mm. pressure.

$$1{\cdot}293\ \text{gm.} = \text{litre by } 760\ \text{mm.},$$

$$\therefore\quad \frac{1{\cdot}293}{760}\ \text{gm.} = \text{litre by mm.}$$

$$73\ \text{litres by } 1{,}000\ \text{mm.},$$

$$\therefore\quad \frac{73 \times 1000 \times 1{\cdot}293}{760}\ \text{gms.},$$

$$i.e.,\quad 124{\cdot}2\ \text{gms.}$$

Ex. 2. A certain quantity of air forms a small spherical bubble of a given radius, when 5 feet below the surface of water; at what depth would the same quantity of air form a bubble of half the given radius, the change of temperature being neglected.

Take the quantity of air as unit of mass **M**, and the first sphere as unit of volume **V**. The pressure of the atmosphere is equivalent to 30 feet of water.

$$1\ \mathsf{M} = \mathsf{V}\ \text{by}\ (30 + 5)\ \text{feet of water},$$

$$(\tfrac{1}{2})^3\ \mathsf{V},$$

$$\therefore\quad \tfrac{1}{8}\ \mathsf{M} = 35\ \text{feet of water},$$

$$1\ \mathsf{M},$$

$$\therefore\quad 35 \times 8\ \text{feet of water},$$

$$i.e.,\quad 280\ \text{feet of water},$$

$$\therefore\quad 250\ \text{feet of depth}.$$

Ex. 3. The content of the receiver of an air pump is 6 times that of the barrel. Find the pressure of the air in the receiver at

the end of the 8th stroke of the piston, the atmospheric pressure being 15 lbs. to the square inch.

Take the original mass of air for unit of mass, and the volume of the receiver for unit of volume ; then

$$1\,\mathbf{M} = \mathbf{V} \text{ by } 15 \text{ lb. wt. per sq. inch.}$$

At the end of the operations we have the same volume ; hence,

$$1\,\mathbf{M} = 15 \text{ lb. wt. per sq. inch.}$$

The volume of the barrel is one seventh that of the receiver and barrel conjointly. As the air will always distribute itself with uniform density, one seventh of the mass will be removed by the first double stroke, one seventh of the remainder by the second double stroke, and so on ; hence after 8 strokes $(1 - \frac{1}{7})^8\,\mathbf{M}$ will be left. But

$$1\,\mathbf{M} = 15 \text{ lb. wt. per sq. inch,}$$

$$\therefore \quad (1 - \tfrac{1}{7})^8 \times 15 \text{ lb. wt. per sq. inch.}$$

$$\begin{array}{rr} \log 6 = \cdot 77815 & \bar{1}\cdot 46440 \\ \log 7 = \cdot 84510 & 1\cdot 17609 \\ \hline \bar{1}\cdot 93305 & 0\cdot 64049 \\ 8 & \\ \hline \bar{1}\cdot 46440 & \end{array}$$

Answer—4·37 lb. wt. per sq. inch.

EXERCISE XXXII.

1. What is the height of the homogeneous atmosphere, when the mercurial barometer is at 30 inches? The specific gravity of air at that pressure is ·00125.

2. In a tube of uniform bore a quantity of air is enclosed. What will be the length of this column of air under a pressure of three atmospheres, and what under a pressure of a third of an atmosphere ; its length under the pressure of a single atmosphere being 12 inches?

3. When the height of the mercurial barometer changes from 29·55 inches to 30·33 inches, what is the change in the mass of 1,000 cubic inches of air, assuming that 100 cubic inches of air weigh 31 grains at the former pressure, and that the temperature remains constantly at 0° C.?

4. A cylindrical bell 4 feet deep, whose content is 20 cubic feet, is lowered into water until its top is 14 feet below the surface of the water, and the air is forced

into it until it is three quarters full. What volume would the air occupy under the atmospheric pressure, the water-barometer being at 34 feet?

5. If the water-barometer stand at 33 feet, to what depth must a diving bell be sunk to reduce the contained air to one-third of its original volume, the height of the bell itself being neglected?

6. A diving-bell is lowered into water at a uniform rate, and air is supplied by a force-pump so as to keep the bell full, without allowing any to escape. How must the rate at which the air is supplied be varied as the bell descends?

7. An air-bubble at the bottom of a pond 10 feet deep, has a volume of 0·00006 of a cubic inch. Find what its volume becomes when it just reaches the surface, the barometer standing at 30 inches.

8. A closed indiarubber ball containing air has a volume of 4 cubic inches at a depth of 100 feet below the surface of water, whose density is unity. If the height of the water-barometer be 30 feet, determine the volume of the ball at the surface of the water, assuming the temperature to remain constant?

9. A Mariotte's tube has a uniform section of 1 square inch, and is graduated in inches; 6 cubic inches are inclosed in the shorter (closed) limb, when the mercury is at the same level in both. What volume of mercury must be poured into the longer limb in order to compress the air into two inches? The barometer stands at 30 inches.

10. Ten cubic centimetres of air are measured off at atmospheric pressure. When introduced into the vacuum of a barometer they depress the mercury which previously stood at 76 centimetres, and occupy a volume of 15 cc. By how much has the mercurial column been depressed?

11. A cylindrical tube, 2 feet long, closed at one end, is lowered down into the sea 200 feet, open end downward like a diving-bell. The atmospheric pressure at the surface being 30 inches of mercury, find how high the water rises in the tube. A column of about $32\frac{1}{2}$ feet of sea-water is equal in weight to a similar column of mercury of 30 inches.

12. If the pressure inside the receiver of an air-pump were reduced to $\frac{1}{3}$ of the atmospheric pressure in 4 strokes, to what would it be reduced in 6 strokes?

13. The cylinder of a single-barrelled air-pump has a sectional area of one square inch, and the length of the stroke is 4 inches. The pump is attached to a receiver whose capacity is 36 cubic inches. Compare the pressure of the air inside the cylinder, after 8 complete strokes of the pump, with the pressure before commencing the operation.

14. If the volume of the cylinder of an air-pump be $\frac{1}{10}$ that of the receiver, find the density of the air in the latter at the end of the fifth stroke.

15. A receiver attached to an air-pump has the volume of 100 cubic inches, while the cylinder has the volume of 10 cubic inches. What proportion of the original air will be left in the receiver after the completion of the fourth double stroke?

16. If the barrel of the common water-pump be 3 feet long, and the tube, supposed of the same cross-section, be 16 feet long; find how high the water will rise after the first stroke, the water-barometer being at 34 feet.

SECTION XXXIII.—WORK.

ART. 156.—**Absolute Units of Work.** Work is done when resistance is overcome; the quantity of work done is proportional to the resisting force and to the distance through which it is overcome. Let the unit of work be denoted by **W**, then

1 **W** = **F** resistance by **L** displacement.

By "displacement" is meant the displacement in the direction of the force.

The British absolute unit is denominated the foot-poundal, and is defined by

1 foot-poundal = poundal resistance by foot displacement.

The C.G.S. absolute unit is denominated the erg, and is defined by

1 erg = dyne resistance by cm. displacement.

ART. 157.—**Gravitation Units.** Work is also measured in terms of gravitation units, by taking the corresponding gravitation unit of force instead of the absolute unit of force. The principal British unit is the foot-pound, defined by

1 foot-pound = lb. by weight by foot displacement.

A metric unit is the kilogrammetre, defined similarly,

1 kilogrammetre = kilogramme by weight by metre.

The principal C.G.S. unit is the gramme-centimetre, defined similarly.

ART. 158.—**Work done by a Pressure.** Suppose that the resistance is a pressure uniformly distributed over the surface of application. Then

1 **F** resistance = (**F** per **S**) by **S** surface,

and 1 **W** = (**F** per **S**) by **S** surface by **L** displacement.

If 1 **S** = $\mathbf{L}^2$, then, since the displacement is normal to the surface of application,

1 **W** = (**F** per $\mathbf{L}^2$) by $\mathbf{L}^3$ volume displacement.

EXAMPLES.

Ex. 1. Reduce 1 foot-pound to ergs, taking the acceleration due to gravity at 981 cm. per second per second.

1 foot-pound = lb. by weight by foot,

1 lb. = 453·6 gm.,

weight = 981 cm. per sec. per sec.,

1 foot = 30·48 cm.,

∴ 1 foot-pound = $981 \times 453·6 \times 30·48$ gm. by cm. per sec. per sec. by cm.

= $981 \times 453·6 \times 30·48$ ergs,

= $1·356 \times 10^7$ ergs.

Ex. 2. A train of 120 tons runs on a level road, and the resistances to be overcome are 8 lbs. per ton. How many units of work must be expended in making a run of 40 miles, when there is no useless expenditure of steam.

8 lb. by weight, resistance = ton of mass,

120 tons, ∴ 120×8 lb. by weight, resistance.

1 foot-pound = lb. by weight by foot advance,

∴ 120×8 foot-pound = foot advance,

$40 \times 5,280$ feet;

∴ $120 \times 8 \times 40 \times 5,280$ foot-pounds,

i.e., $2·03 \times 10^8$,,

Ex. 3. How many units of work must be expended in raising from the ground the materials for building a uniform column 66 feet 8 inches high and 21 feet square, a cubic foot of brickwork weighing one hundredweight.

As the column is uniform there is the same amount of matter in the different courses, but the height through which the matter of a course is raised increases uniformly from the bottom to the top. Hence the correct result will be got by taking the average height.

112 lbs. = cubic foot,

= square foot by foot height,

$$21^2 \text{ square feet by } \frac{200}{3} \text{ feet height;}$$

$$\therefore \qquad 112 \times 21^2 \times \frac{200}{3} \text{ lbs.}$$

The average height is $\frac{100}{3}$ feet,

$$\therefore \qquad \frac{112 \times 21^2 \times 200 \times 100}{9} \text{ foot-pounds,}$$

$$i.e, \qquad 1{\cdot}098 \times 10^8 \qquad\qquad ,,$$

Ex. 4. What is the amount of work done per stroke by an engine when the average pressure of the steam during the stroke is 38 lb. weight per square inch, the length of the stroke being 3 feet, and the diameter of the piston 14 inches;

1 foot-pound = lb. wt. per sq. in. by sq. in. by ft.

38 lb. wt. per sq. in. by $7^2\pi$ sq. in. by 3 ft. per stroke,

$$\therefore \qquad 38 \times 7^2 \times 3\pi \text{ foot-pounds per stroke,}$$

$$i.e, \qquad 17{,}556 \qquad\qquad ,,$$

EXERCISE XXXIII.

1. Reduce a kilogrammetre to foot-pounds, and reciprocally.

2. The resistance to traction on a level road is 150 lbs. per ton moved; how many foot-pounds of work are expended in drawing 6 tons through a distance of 150 yards?

3. A hole is punched through a plate of wrought-iron one half inch in thickness, and the pressure actuating the punch is estimated at 36 tons. Assuming that the resistance to the punch is uniform, find the number of foot-pounds of work done.

4. It is found, neglecting friction, that a force acting horizontally will move 10 lbs. up 5 feet of an incline rising 1 in 4. Find the work done, and find also the force parallel to the plane which will just support the weight of 10 lbs.

5. Calculate the amount of work done, independently of that lost through friction, in drawing a car of two tons weight, laden with 30 passengers averaging 11 stones each in weight, up a slope, the ends of which differ in level by 50 feet.

6. The inclination of a mountain path to the horizon is 30°; how much work is done against gravity by a man of 12 stones weight in walking a mile along the path?

7. How much work is done by means of a crane in raising from the ground the material required to build a stone wall 100 feet long, 36 feet high, and 2 feet thick, the density of the stone being 153 pounds per cubic foot.

8. A fly-wheel weighing 7 tons turns on a horizontal axle 1 foot in diameter. If the coefficient of friction between the axle and its bearing is 0·075, what number of foot-pounds of work must be done against friction while the wheel makes 10 turns?

9. The resistance of friction along an inclined plane is taken at 150 lbs. for each ton of weight moved. Find the work done in drawing 2 tons up 100 feet of an incline which rises 1 foot in height for 25 in length.

10. Weights of 10 lbs. and 8 lbs. are connected by a very fine thread which rests on a rough fixed pulley, so that they hang suspended; the heavier weight is found to be just not heavy enough to fall and draw the lighter weight up; if now we suppose the weights to move uniformly, so that one goes up and the other down through 12 feet, how many foot-pounds of work are done against friction during the motion?

11. The plunger of a force-pump is $8\frac{3}{4}$ inches in diameter, the length of the stroke is 2 feet 6 inches, and the pressure of the water is 50 lbs. per square inch; find the number of units of work done in one stroke.

12. Determine the unit of mass in order that the absolute unit of work may be equal to the foot-pound, the second and foot being the units of time and length, and 32·2 feet per sec. per sec. being the implied intensity of gravity.

SECTION XXXIV.—KINETIC ENERGY.

ART. 159.—**Kinetic Energy**. By kinetic energy is meant the equivalent of the work spent in increasing the speed of a body. The work spent is proportional to the mass of the body, to its initial speed, and to the increment of speed. It is evident that the increment of speed must be taken small, in order that the initial speed may not vary sensibly. Hence

$$1\ \mathsf{W} = \mathsf{M} \text{ by } (\mathsf{L} \text{ per } \mathsf{T}) \text{ initial by } (\mathsf{L} \text{ per } \mathsf{T}) \text{ increment.}$$

To find the entire amount of kinetic energy in a body, we have to consider that the total increment is the final speed, and that the average initial speed is half the final speed; hence

$$\tfrac{1}{2}\ \mathsf{W} = \mathsf{M} \text{ by } (\mathsf{L} \text{ per } \mathsf{T} \text{ final})^2.$$

As particular cases

$$\tfrac{1}{2} \text{ foot-poundal} = \text{lb. by (foot per second)}^2,$$

$$\tfrac{1}{2} \text{ erg} = \text{gm. by (cm. per second)}^2.$$

Observe that the dimensions of each of the equivalents of **W** are the same.

ART. 160.—**Kinetic Energy of Rotation ; Moment of Inertia.** We have

$$\tfrac{1}{2}\,\mathsf{W} = \mathsf{M} \text{ by } (\mathsf{L} \text{ per } \mathsf{T})^2.$$

Suppose that a rigid body revolves round an axis, and that the whole of the mass of the body is at nearly the same distance from the axis ; then since

$$1 \ \mathsf{L} \text{ per } \mathsf{T} = \mathsf{L} \text{ radius by (radian per } \mathsf{T}),$$

it follows that

$$\tfrac{1}{2}\,\mathsf{W} = \mathsf{M} \text{ by } (\mathsf{L} \text{ radius})^2 \text{ by (radian per } \mathsf{T})^2,$$

$$\text{or } \tfrac{1}{2}\,\mathsf{W} \text{ per (radian per } \mathsf{T})^2 = \mathsf{M} \text{ by } (\mathsf{L} \text{ radius})^2.$$

The idea **M** by (**L** radius)^2 depends on the body only ; it is called the *moment of inertia* of the body. From the above the unit is seen to be equivalent to half unit of work per square of unit of angular velocity.

ART. 161.—**Radius of Gyration.** When the several portions of the body are at different distances from the axis, an equivalent radius can be found such that, were the whole mass situated at its end, the value of **M** by **L**^{2} would be the same as the sum of the actual values of **M** by **L**^{2} for the different portions. This equivalent radius is called the *radius of gyration*. Compare Arts. 31 and 136.

RADIUS OF GYRATION.

Bar revolving about one end, - - -	·577 **L** per **L** length.
Bar revolving about its centre, - - -	·289 ,, ,,
Circular plate revolving about its centre, -	·707 **L** per **L** radius.
Circular plate revolving about its diameter, -	·5 ,, ,,
Sphere about its diameter, - - - -	·6324 ,, ,,
Thin hollow sphere about its diameter, -	·816 ,, ,,
Cylinder revolving about its axis, - - -	·707 ,, ,,
Cone revolving about its axis, - - -	·548 **L** per **L** radius of base.

EXAMPLES.

Ex. 1. A locomotive weighing 30 tons is moving with a speed of 60 miles an hour. Express its kinetic energy in foot-pounds, and also in foot-poundals.

$$\tfrac{1}{2} \text{ foot-poundal} = \text{lb. by (ft. per sec.)}^2,$$
$$30 \times 20 \times 112 \text{ lb. by } (88)^2 \text{ (ft. per sec.)}^2;$$
$$\therefore \quad \tfrac{1}{2} \times 30 \times 20 \times 112 \times 88^2 \text{ foot-poundals},$$
$$\textit{i.e.,} \quad 2{\cdot}602 \times 10^8 \text{ foot-poundals}.$$

But
$$1 \text{ foot-pound} = 32{\cdot}2 \text{ foot-poundals},$$
$$\therefore \quad \frac{2{\cdot}602 \times 10^8}{32{\cdot}2} \text{ foot-pounds},$$
$$\textit{i.e.,} \quad 8{\cdot}08 \times 10^6 \text{ foot-pounds}.$$

Ex. 2. A body, whose mass is 100 gms., is thrown vertically upwards with a velocity of 980 cm. per sec. What is the kinetic energy of the body, *first*, at the moment of propulsion; *second*, after half a second; *third*, after one second?

$$\tfrac{1}{2} \text{ erg} = \text{gm. by (cm. per sec.)}^2,$$
$$100 \text{ gm.}$$
$$\therefore \quad 50 \text{ erg} = \text{(cm. per sec.)}^2.$$

First, at the moment of propulsion the speed is 980 cm. per sec., $\therefore$ the energy is

$$50 \times 980^2 \text{ ergs, } \textit{i.e.,} \ 4{\cdot}802 \times 10^7 \text{ ergs, or 48 megalergs.}$$

Second, after half a second the speed is 490 cm. per sec., for gravity in that time has deducted 490 cm. per sec.; hence the energy is

$$50 \times 490^2 \text{ ergs, } \textit{i.e.,} \ 1{\cdot}2005 \times 10^7 \text{ ergs.}$$

Third, after one second the speed has been reduced to zero; $\therefore$ the energy has also been reduced to zero.

Ex. 3. Find the number of units of work required to start on a level a laden tram car of 4 tons so as to give it a speed of 6 miles an hour; and also the average force which must be applied in order that the start may be made in two seconds.

$$\tfrac{1}{2} \text{ foot-poundal} = \text{lb. by (ft. per sec)}^2,$$

$$4 \times 112 \times 20 \text{ lb. by } \left(\frac{44}{5}\right)^2 \text{(ft. per sec.)}^2;$$

$$\therefore \qquad \frac{4 \times 112 \times 20 \times 44^2}{2 \times 25} \text{foot-poundals,}$$

i.e., 346931 foot-poundals.

To find the force.

$$1 \text{ poundal} = \text{(lb. by ft. per sec.) per sec.,}$$

$$\frac{4 \times 112 \times 20 \times 44}{5} \text{(lb. by ft. per sec.) per 2 sec.;}$$

$$\therefore \qquad 4 \times 112 \times 2 \times 44 \text{ poundals,}$$

i.e., 39424 poundals.

Ex. 4. What average force will bring to rest in 100 ft. a train of 30 tons which has a speed of 10 miles an hour. Also, what average force will bring it to rest in 5 seconds ?

$$\tfrac{1}{2} \text{ foot-poundal} = \text{lb. by (ft. per sec.)}^2,$$

$$30 \times 20 \times 112 \text{ lb. by } \left(\frac{44}{3}\right)^2 \text{(ft. per sec.)}^2;$$

$$\therefore \qquad \frac{30 \times 20 \times 112 \times 44^2}{2 \times 9} \text{foot-poundals ;}$$

but $\quad 1 \text{ foot-poundal per ft. advance} = \text{poundal},$

$$\frac{30 \times 20 \times 112 \times 44^2}{2 \times 9} \text{ft. poundals per 100 ft.;}$$

$$\therefore \qquad \frac{30 \times 20 \times 112 \times 44^2}{2 \times 9 \times 100} \text{poundals,}$$

i.e., 72277 poundals.

Again, $\quad \dfrac{30 \times 20 \times 112 \times 44}{3} \text{lb. by ft. per sec.} = 5 \text{ secs.,}$

$$\therefore \qquad \frac{30 \times 20 \times 112 \times 44}{3 \times 5} \text{lb. by ft. per sec. per sec.,}$$

$$\therefore \qquad 197120 \text{ poundals.}$$

Ex. 5. What is the moment of inertia of a grindstone, 18 inches in diameter, and 4 inches in thickness, the density being ·09 lb. per cubic inch ?

18^2 (inch diam.)^2 by 4 inch thick,

$\frac{\pi}{4}$ cubic inch = (inch diam.)^2 by inch thick,

$\cdot 09$ lb. = cubic inch,

$\therefore \quad \cdot 09 \times 18^2 \pi$ lb.

Now $\frac{1}{2}$ (ft. radius of gyration)^2 = (ft. radius of disc.)^2,

($\frac{3}{2}$ ft. radius of disc)^2,

$\therefore \quad \frac{9}{8}$ ft.^{2} (ft. radius of gyration)^2;

$$\therefore \quad \frac{\cdot 09 \times 18^2 \times 9 \times \pi}{8} \text{ lb. by ft.}^2.$$

i.e., 103 „

Ex. 6. What is the kinetic energy of the grindstone when revolving at the rate of 1·5 revolutions per second?

1·5 rev. = sec.,

2π radian = rev.,

$\therefore \quad 3\pi$ radian = sec.;

$$\therefore \quad \frac{\cdot 09 \times 18^2 \times 9 \times \pi \times 9 \times \pi^2}{8} \text{ lb. by ft.}^2 \text{ by (radian per sec.)}^2,$$

i.e., 7271 foot-poundals.

EXERCISE XXXIV.

1. A mass of 6 cwt. moves with a velocity of 20 feet per second; how many units of work are stored up in it?

2. Compare the amounts of kinetic energy in a pillow of 20 lbs. which has fallen through one foot vertically, and an ounce bullet moving at 200 feet per second.

3. Find the number of units of work accumulated in a 24-lb. shot leaving the mouth of the gun with a velocity of 1,500 feet per second.

4. A mass of snow, 28 lbs. in weight, falls from the roof of a house to the ground, a distance of 40 feet. Calculate the kinetic energy acquired at the time of impact.

5. A ball weighing five ounces, and moving with a velocity of 1,000 feet per second, strikes a shield, and after piercing it moves on with a velocity of 400 feet per second. How much energy has been expended in piercing the shield?

6. Calculate the kinetic energy of a tram-car weighing 2·5 tons, when it is moving at the rate of 6 miles an hour, and is laden with 36 passengers, averaging 11 stones each in weight.

7. If the coefficient of kinetic friction for a tram-car moving on its rails is $\frac{1}{8}$, find how much work is done when the above car, loaded as stated, is pulled 3 miles along a level road.

8. Calculate the kinetic energy of a hammer of 5 tons let fall half a foot.

9. Two 30-ton hammers moving in opposite directions at 20 feet per second, simultaneously strike a mass of soft iron. How many foot-pounds of work will they do upon it?

10. A 32-lb. ball is thrown vertically upwards with a velocity of 20 feet per second. What is its kinetic energy when it has gone 5 feet?

11. A train of 200 tons starting from rest acquires a velocity of 40 miles an hour in 3 minutes on a horizontal railroad. What is the excess of the moving above the retarding forces, each being assumed uniform?

12. What is the amount of kinetic energy in an engine of 25 tons when moving with a velocity of 20 miles per hour? What force, measured in poundals, acting for ten seconds, is sufficient to stop the engine?

13. What average force will bring to rest in 20 feet a tram-car of 5 tons, having a velocity of 6 miles per hour?

14. Supposing the coefficient of friction to be 0·05, how far will a railway carriage run on level rails with a velocity of 10 miles an hour?

15. If an ounce bullet leaves a gun with a velocity of 800 feet per second, the gun barrel being 3 feet long, what would be the accelerating force on the bullet, supposing it to have been acting uniformly throughout?

16. A bullet weighing $2\frac{1}{2}$ ounces leaves a gun with a velocity of 1,550 feet per second; the length of the gun barrel is $2\frac{1}{2}$ feet; find the average accelerating force upon the bullet within the barrel, and express it in gravitation units.

17. A shot of 1,000 lbs., moving at 1,600 feet per second, strikes a fixed target; how far will the shot penetrate the target, exerting upon it an average pressure equal to the weight of 12,000 tons?

18. A locomotive of 15 tons, being supposed to acquire a speed of 20 miles an hour in moving through a mile of distance, under the action of a constant difference of moving and resisting forces; calculate in lb. weight the requisite difference of the forces.

19. A heavy body is projected up an incline rising 1 in 100; the friction against the plane is one tenth of the pressure; find the distance it will travel before being reduced to rest, the velocity of projection being 121 feet per second.

20. Find the tension on a rope which draws a carriage of 8 tons up a smooth incline of 1 in 5, and causes an increase of velocity of 3 feet per second per second.

21. If on the same incline the rope breaks when the carriage has a velocity of 48·3 feet per second, how far will the carriage continue to move up the incline?

22. A mass P, after falling freely through h feet, begins to pull up a heavier mass Q by means of a string passing over a pulley as in Attwood's machine; find the height through which Q will be lifted.

23. Two masses of 5 and 10 lbs. respectively impinge directly, moving with velocities of 8 and 10 feet per second. Find the common velocity after impact, and show that there has been a transformation of kinetic energy.

24. A bullet weighing 50 grammes is fired into a target with a velocity of 500 metres a second. The target is supposed to weigh a kilogramme, and to be free to move. Find in kilogrammetres the loss of energy in the impact.

25. An ounce-bullet leaves the mouth of a rifle with a velocity of 1,500 ft. per sec. If the barrel be 4 ft. long, calculate the mean pressure of the powder, neglecting all friction.

26. The bob of a simple pendulum is pulled through an arc of 60 degrees, and let go. Compare its kinetic energy after describing an arc of 30 degrees with its energy at the lowest point.

27. Find the initial velocity of a shot of 1,000 lbs. discharged from a 100-ton gun, supposing none of the 30,000 foot-tons of energy given out by the explosion to be wasted in heat, light, or sound.

28. A cannon-ball of 5 kgm. is discharged with a velocity of 500 metres per second; find its kinetic energy in ergs. If the cannon be freely suspended, and have a mass of 100 kgm., find in ergs the energy of the recoil.

29. The moment of inertia of a ring is 50 kilogramme by $(\text{metre})^2$; what is it in terms of pound by $(\text{foot})^2$?

30. A fly-wheel has a mass of 30 tons, which may be supposed to be distributed along the circumference of a circle 8 feet in radius; it makes 20 revolutions per minute; find its kinetic energy in foot-pounds.

31. What is the kinetic energy of a circular saw having a diameter of 2 feet, and $\frac{1}{8}$ inch thick, when moving with a circumferential velocity of 6,000 feet per minute? The density of steel is 500 lb. per cubic foot.

32. The rim of a fly-wheel, specific gravity 7·75, whose inner and outer radii are 4 and 5 feet respectively, and whose thickness is 1 foot, revolves uniformly 20 times per minute round its axis; calculate in foot-pounds the entire amount of work accumulated in it.

33. A cord, which may be taken as weightless, is wrapped round the circumference of a wheel of 3 feet radius, and a weight of 14 lbs. is attached to the free end of the cord. The mass of the wheel is 300 lbs., and its radius of gyration about the axis is $2\frac{1}{2}$ feet. The weight being let go from rest falls for 2 seconds; find how far it has fallen, and its velocity at the end of that time. There is supposed to be no friction.

34. A mass of 10 lbs. is attached by a string to the rim of a circular disc, the mass of which is 25 lbs., and the radius of gyration $1\frac{1}{2}$ feet. Find the angular velocity of the disc 5 seconds after the weight is allowed to fall, supposing that there is no friction.

35. A uniform circular cylinder of 100 lbs. mass and one foot radius revolves frictionlessly on its axis, which is horizontal. A thread rolled round the cylinder

carries on one end, which hangs down freely, a weight of one lb. Find how far the weight falls from rest in 3 seconds.

36. A rod of uniform density, which can turn freely round one end, is let fall from a horizontal position ; what is its angular velocity when it reaches its lowest position ?

SECTION XXXV.—POWER.

ART. 162.—**Power.** By power is meant the rate at which work is done by an agent. It is expressed in terms of **W** per **T**. The term *activity* is used by Sir W. Thomson and Professor Tait to denote this idea.

The British absolute unit is the foot-poundal per second, and the C.G.S. unit is the erg per second. Gravitation units are foot-pound per sec., kilogrammetre per minute, etc.

A practical unit is the *horse-power*, founded by Watt on an estimate of the average rate at which a horse can work ;

1 horse-power = 33,000 foot-pounds per minute,
= 550 foot-pounds per second.

The intensity of gravity is taken at its standard value.

The corresponding French unit is the *force de cheval ;*

1 force de cheval = 75 kilogrammetres per second.

It has recently been proposed to introduce the term "watt" to denote 10^7 erg per sec., which is a convenient unit of power in electrical measurements.

EXAMPLES.

Ex. 1. Calculate the amount of work done against gravity in drawing a car of 2·5 tons, laden with 30 passengers averaging 11 stones each, up an incline, the ends of which differ in level by 120 feet. Also, find the horse-power sufficient to do that work in half an hour.

$(2{\cdot}5 \times 20 \times 112 + 30 \times 11 \times 14)$ lb. weight, and 120 feet height,

$\therefore \quad (2{\cdot}5 \times 20 \times 112 + 30 \times 11 \times 14)\ 120$ foot-pounds,

i.e., 1,226,400 foot-pounds.

1 horse-power = 33,000 foot-pounds per minute,

30 minutes,

$\therefore$ 1 horse-power = $33{,}000 \times 30$ foot-pounds;

$$\therefore \quad \frac{30 \times 33{,}000}{1{,}226{,}400} \text{ horse-power,}$$

i.e., 1·24 horse-power.

Ex. 2. Express a horse-power in terms of the C.G.S. absolute unit.

33,000 foot-pounds per minute,

i.e., 33,000 lb. by weight by foot per minute.

Now $\frac{5{,}000}{11}$ gm. = lb.,

981 cm. per sec. per sec. = weight,

$\frac{3{,}200}{105}$ cm. = ft.,

60 sec. = min.;

$$\therefore \quad \frac{5{,}000 \times 981 \times 3{,}200}{11 \times 105 \times 60} \text{ gm. by cm. per sec. per sec. by cm. per sec.,}$$

= lb. by weight by ft. per min.;

$$\therefore \quad \frac{33{,}000 \times 5{,}000 \times 981 \times 3{,}200}{11 \times 105 \times 60} \text{ ergs per sec.,}$$

i.e., $7{\cdot}47 \times 10^9$ erg per sec.

4·51851	1·04139
3·69897	2·02119
2·99167	1·77815
3·50515	———
———	4·84073
14·71430	
4·84073	
———	
9·87357	

EXERCISE XXXV.

1. The French "force de cheval" is 4,500 kilogrammetres per minute; express it in terms of the horse-power.

2. Given 1 horse-power = 550 foot-pounds per sec., and 1 "force de cheval" = 75 kilogrammetres per sec.; deduce the relation of the "force de cheval" to the horse-power. It is supposed that the intensity of gravity is the same in the two definitions.

3. Find the value of the horse-power in terms of kilogrammetres per minute.

4. Determine the horse-power of a machine capable of raising 10 tons through a height of 20 feet in 2 minutes.

5. If an engine consumes 2 pounds of coal per horse-power per hour, how many foot-pounds of work will it perform when consuming 112 pounds of coal?

6. How many foot-pounds of work are required to raise 30,000 lbs. of water from a depth of a furlong; and how many horse-power to do it in five minutes?

7. If a pressure of 1 ton is exerted through 10 yards, how many foot-pounds of work are done; and at what horse-power does an engine work which does the work in half a minute?

8. A pumping engine is partly worked by a weight of 2 tons, which at each stroke of the pump falls through 4 ft.; the pump makes 10 strokes a minute; how many gallons of water are lifted per minute by the weight from a depth of 200 ft.?

9. Calculate the horse-power of an engine from the following data:—stroke 24 in., diameter of piston 16 in., 100 revolutions per minute, average effective pressure in cylinder 60 lbs. per square inch.

10. In the transmission of power by a rope, the wheel carrying the rope is 14 feet in diameter and makes 30 revolutions per minute, the tension of the rope being 100 lbs. Find the amount of power transmitted estimated as horse-power.

11. What diameter of cylinder will develop 50 horse-power with a four-foot stroke, 40 revolutions per minute, and a mean effective steam pressure of 30 lbs. per square inch above the atmosphere, the engine being non-condensing?

12. The cylinder of an engine is 12 inches diameter by 20 inches long; with an average pressure of 60 lbs. per square inch it has a power of 40 horse-power. Find the rate of revolution of the engine.

SECTION XXXVI.—MECHANICAL ADVANTAGE.

ART. 163.—Virtual Velocity.

Since $\qquad 1\ \mathsf{W} = \mathsf{F}$ resistance by L displacement,

$1\ \mathsf{W}$ per $\mathsf{T} = (\mathsf{F}$ resistance by L displt.) per T,

$= \mathsf{F}$ resistance by (L displt. per T);

for the force is supposed to be constant during the displacement. The velocity indicated is that of the force in its own direction, that is, the *virtual velocity*.

ART. 164.—**Velocity Ratio and Mechanical Advantage.** The scholium to Newton's third law of motion asserts that in the case of any machine, the rate at which work is done by the agent is equal to the rate at which work is done against the resistance, when there is no acceleration of the parts of the machine, and no loss of energy through friction.

Suppose that the speed of the agent is v **L** per **T**, and that of the resistance v' **L** per **T**; then

$$v/v' \ \mathsf{L} \text{ displt. of agent} = \mathsf{L} \text{ displt. of resistance.}$$

But, as the value of **F** by **L** is the same for the agent and for the resistance,

$$v'/v \ \mathsf{F} \text{ agent} = \mathsf{F} \text{ resistance.}$$

The former is called the *velocity ratio* of the machine, and the latter the *mechanical advantage*. The value of the one is the reciprocal of the value of the other.

EXAMPLES.

Ex. 1. If the thread of a screw makes 25 turns in 3 inches, and the arm is 24 inches; what force must be applied to sustain a weight of 1 cwt.?

$$2\pi \times 24 \text{ inch, agent} = \text{turn of screw,}$$

$$25 \text{ turns of screw} = 3 \text{ inch, resistance;}$$

$$\therefore \quad 2\pi \times 24 \times 25 \text{ inch, agent} = 3 \text{ inch, resistance;}$$

$$\therefore \quad \frac{3}{2\pi \times 24 \times 25} \text{ lb. wt., agent} = \text{lb. wt. resistance;}$$

$$112 \text{ lb. wt., resistance,}$$

$$\therefore \quad \frac{3 \times 112}{2\pi \times 24 \times 25} \text{ lb. wt., agent,}$$

$$i.e. \quad \cdot 089 \text{ lb. wt., agent.}$$

Ex. 2. In a crane the numbers of the teeth in the two wheels are 45 and 60 respectively; the numbers of leaves in the two pinions 9 and 15; the arm of the winch is 3 feet, and the radius of the weight-bearing axle $4\frac{1}{2}$ inches. Find the mechanical advantage of the crane.

$$6\pi \text{ ft., agent} = \text{revolution of 1st pinion,}$$
$$45 \text{ revn. of 1st pinion} = 9 \text{ revn. of 1st wheel,}$$
$$1 \text{ revn. of 1st wheel} = \text{revn. of 2nd pinion,}$$
$$60 \text{ revn. of 2nd pinion} = 15 \text{ revn. of 2nd wheel,}$$
$$1 \text{ revn. of 2nd wheel} = \frac{2\pi \times 9}{2 \times 12} \text{ ft., resistance;}$$
$$\therefore \frac{6\pi \times 45 \times 60 \times 12 \times 2}{9 \times 15 \times 9 \times 2\pi} \text{ ft., agent} = \text{ft., resistance,}$$
$$i.e, \quad 160 \text{ ft., agent} = \text{ft., resistance,}$$
$$\therefore \quad 1/160 \ \mathsf{F} \text{ agent} = \mathsf{F} \text{ resistance:}$$

Ex. 3. In a small hydraulic press the ram is 2 in. and the plunger $\frac{1}{2}$ in. in diameter; the length of the lever handle is 2 ft., and the distance from the fulcrum to the plunger is $1\frac{1}{2}$ in. Find the power exerted on the ram when a 10-lb. weight is hung at the end of the lever.

$$\tfrac{1}{16} \text{ sq. inch plunger} = \text{sq. inch ram;}$$
$$\therefore \quad 16 \text{ in. displacement of plunger} = \text{in. displacement of ram,}$$
$$\text{and} \quad 24 \text{ in. displacement of agent} = \tfrac{3}{2} \text{ in. displacement of plunger;}$$
$$\therefore \frac{24 \times 16 \times 2}{3} \text{ in. displacement of agent} = \text{in. displacement of ram;}$$
$$\therefore \quad 1 \text{ lb. wt. agent} = 256 \text{ lb. wt. ram,}$$
$$10 \text{ lb. wt. agent;}$$
$$\therefore \quad 2{,}560 \text{ lb. wt. ram.}$$

EXERCISE XXXVI.

1. Find the steepest incline up which a force of 5 cwt. can just move a weight of 2 tons, friction being left out of account.

2. A man weighing 12·5 stones is lowered into a well by means of a windlass, the arm and axle of which are 30 inches and 8 in diameter. Find the force which must be applied to let him down with uniform velocity.

3. The wire-rope working a railway signal passes over and has its end attached to the rim of a fixed grooved pulley; the pulley is 1 foot in diameter, and is turned by a lever 3 feet long. Find the tension transmitted along the rope when a man pulls with a force of 37 lbs. at right angles to the lever and at its extremity.

4. Twelve sailors, each exerting a force of 48 pounds, work a capstan with levers 7 feet 6 inches long; the radius of the capstan is 16 inches; what resistance can they unitedly sustain?

5. A driving wheel has 64 teeth, and the driven wheel 33 teeth; how many revolutions per second will the latter make when the former makes 10 revolutions per second?

6. A screw having $4\frac{1}{2}$ threads to the inch is worked by an arm 18 inches long; what is the force exerted by the screw when a force of 15 lbs. is applied at the end of the arm?

7. A screw, the pitch of which is a quarter inch, is turned by means of a lever 4 feet long; find the force which will raise 15 cwt.

8. The screw of a steamer has a pitch of p feet, and makes n revolutions per minute; deduce the theoretical velocity of the steamer in knots per hour.

9. If the thread of a screw be inclined at an angle of 30 degrees to the horizontal, the radius of the horizontal section of the cylinder 9 inches and the length of the lever 4 feet; find what power will sustain 15 cwt. on it.

10. In a screw-jack, where a worm wheel is used, the pitch of the screw is $\frac{5}{8}$ inch, the number of teeth in the worm wheel is 16, and the length of the lever handle is 10 inches; find the gain in power.

11. A wheel and axle is used to raise a bucket from a well. The radius of the wheel is 15 inches; and while it makes 7 revolutions the bucket, which weighs 30 lbs., rises $5\frac{1}{2}$ feet. Show what is the smallest force that can be employed to turn the wheel.

12. Suppose that we have four weightless pulleys, three moveable and one fixed, forming an example of the first system, and that the weight is a man weighing 160 lbs.; find what pull the man must exert on the power end of the rope in order to raise himself thereby.

13. A weight of 3 cwt. is raised 3 feet by means of a single moveable pulley, the block of which has three sheaves; determine the power and the space through which it has acted.

14. In the second system of pulleys, if there be three pulleys in the lower block, which weighs 8 lbs., and the string be fastened to the upper block; find the weight which a power of 20 lbs. can support.

15. Find the ratio of the power to the weight in a system of three pulleys, in which all the strings are attached to the weight, neglecting the weight of the pulleys.

16. In a compound wheel and axle the diameters of the two parts of the axle are 5 and 6 inches respectively. The weight raised hangs from a single moveable

pulley in the usual manner, and is supported by a pressure applied perpendicularly to a lever handle 15 inches in length. Find the ratio of the agent to the resistance.

17. In a lifting crab the lever handle is 14 inches long, the diameter of the drum is 6 inches, and the wheel and pinion have 57 and 11 teeth respectively. Find the weight in pounds which could be raised by a force of 50 lbs. applied to the lever handle, friction being neglected.

18. The thread of a screw makes 12 turns in a foot of its length ; the power is applied at the end of an arm 2 feet long ; it is found that when the power is 30 lbs. it can just raise a weight of 1,200 lbs. ; what portion of the power is used in overcoming friction, and how many foot-pounds of work are done by the power when the weight is raised 2 feet ?

SECTION XXXVII.—MOMENT OF A FORCE.

ART. 165.—**Moment of a Force.** When the virtual velocity of an agent is round a circle, F by L per T can be expressed as F by L radius by (radian per T). The angular velocities of the agent and the resistance being the same, the condition that the value of W per T must be the same for both reduces to the condition that the value of F by L radius be the same. This idea F by L radius is called the *moment of a force.* Special units are poundal by foot, lb. by weight by foot, kgm. by weight by metre, etc.

Let r L and r' L be the measures of the arms, then, since the angular velocity is the same for both, the velocity-ratio becomes

$$r/r' \text{ L agent} = \text{L resistance} ;$$

and therefore the mechanical advantage becomes

$$r'/r \text{ F agent} = \text{F resistance}.$$

ART. 166.—**Horizontal Lever.** In the case of a vertical force acting at the end of a horizontal lever, the displacement of the end of the lever is at the beginning in the direction of the force. Hence the condition for equilibrium is that the value of F by L arm should be the same for the two opposing forces. When the

two forces are the weights of masses, as the intensity of gravity is the same for the two, the condition reduces to **M** by **L** arm being the same for the two.

ART. 167.—**Couple.** The resultant of two parallel forces is equal to their sum, and acts at any point round which the moment of the one force is equal to the moment of the other force.

When the two parallel forces are equal and act in opposite directions, there is no resultant. The forces then constitute what is called a *couple*. Their effect is to produce rotation without any translation.

The moment of the couple is the sum of the moments of the two forces about any point in the common arm. Hence its value is given by that of either force and the length of the arm.

Rate at which work is done by a couple,

1 **W** per **T** = **F** force by **L** arm by (radian per **T**).

EXAMPLES.

Ex. 1. Find the value of foot-ton of statical moment in terms of kilogrammetre.

1 ton	= 20 cwt.,
1 cwt.	= 50·8 kgm.,
1 foot	= 12 inches,
39·37 inches	= metre.

Multiply the expressions on either side, and leave out such units as appear on both sides.

39·37 ton by foot = $20 \times 50{\cdot}8 \times 12$ kgm. by metre,

$$\therefore \quad \frac{20 \times 50{\cdot}8 \times 12}{39{\cdot}37} \text{ kgm. by metre} = \text{ton by foot,}$$

i.e., 309·7 „

Ex. 2. Forces of 5 and 7 poundals act in the same direction along parallel lines at points 2 feet apart; where is their centre? If the direction of the former force is reversed, where will the centre be then?

In the first case the resultant must act between the two given forces. Suppose x feet from the 7 poundal force, then it must be $2-x$ feet from the 5 poundal force. Now

$$7 \times x \text{ poundal by foot} = (2-x)\ 5 \text{ poundal by foot}.$$

$$\therefore \quad 7x = 10 - 5x,$$

$$\therefore \quad x = \tfrac{10}{12}.$$

Hence 10 inches from the greater force.

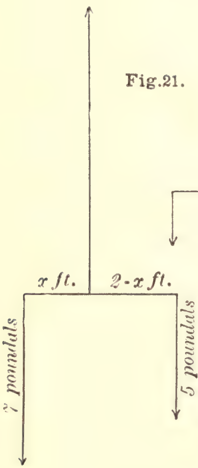


Fig.21.

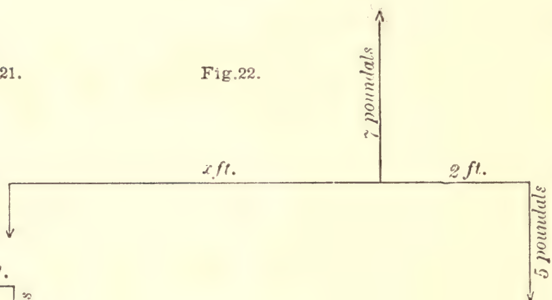


Fig.22.

In the second case the resultant acts outside and on the side of the greater force. Let the measure of its distance from the greater force be x feet. Then

$$x \times 7 \text{ poundal by foot} = (x+2)5 \text{ poundal by foot},$$

$$\therefore \quad x = \frac{10}{2}\ ; \text{ hence 5 feet.}$$

Ex. 3. The arms of a balance being of unequal lengths, a mass of 16 lbs. appears to be only 14 lbs. What would be its apparent mass if weighed in the other scale.

Here we have to consider only lb. by ∟ arm, because the intensity of gravity is the same for the mass and the counterpoise.

14 lb. counterpoise = 16 lb. mass,

∴ 16 ∟ arm of counterpoise = 14 ∟ arm of mass.

But for second weighing

14 L arm of counterpoise = 16 L arm of mass,

$\therefore$ 16 lb. counterpoise = 14 lb. mass,

16 lb. mass,

$\therefore$ $\frac{16 \times 16}{14}$ lb. counterpoise,

i.e., 18·3 lbs.

Ex. 4. A lever, 12 feet long, is fixed at one end and at a point $3\frac{1}{2}$ feet from that end, and carries at the other end a weight of 20 lbs.; find the forces at the two fixed points.

20×12 lb. by weight by foot,

3·5 feet;

$\therefore$ $\frac{20 \times 12}{3 \cdot 5}$ lb. by weight,

i.e., 68·6 ,,

Hence the remaining force is $68 \cdot 6 - 20$ lb. by weight,

i.e., 48·6 pound weight.

Ex. 5. A uniform bar weighs 2·5 gm. per cm. Find its length when a mass of 12·5 gm. suspended at one end keeps it in equilibrium about a fulcrum 12 cm. from the other end.

2·5 gm. per cm.

12 cm. $\therefore$ $12 \times 2 \cdot 5$ gm. in one arm.

Let length of other arm be x cm., then x 2·5 gm. in it.

The weight of each arm acts as if its mass were concentrated at its mid-point, hence the moments are

$$12 \times 2 \cdot 5 \times \frac{12}{2} \text{ gm. by cm.; and } \frac{x^2}{2}\, 2 \cdot 5 \text{ gm. by cm.}$$

But the moment of the suspended mass is

$$12 \cdot 5x \text{ gm. by cm.};$$

hence
$$12 \cdot 5x + \frac{x^2}{2} 2 \cdot 5 = \frac{12^2}{2} 2 \cdot 5\,;$$

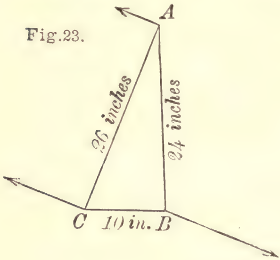
$$x^2 + 10x = 144,$$

$x = 18$, and the whole length is 30 cm.

Ex. 6. ABC is a triangle right-angled at B, AB being 2 feet long, and BC 10 inches. At A a force 1 F is applied at right

angles to AC, at C a force 2 **F** in the same direction, and at B a force 3 **F** in the opposite direction. Find the moment of the resulting couple.

Fig.23.



The third side of the triangle is

$$\sqrt{24^2 + 10^2} \text{ inch};$$

$$i.e., \quad 26 \text{ inch}.$$

Let x inches denote the distance of the centre of the parallel forces from C; then

$$2 \times x \ \mathbf{F} \text{ by inch} = (26 - x) \ \mathbf{F} \text{ by inch},$$

$$\therefore \quad x = \frac{26}{3}.$$

The distance from C of the point of application of the force at B is the component of CB along CA.

10 inches component along CA = 26 inches along CB,

10 inches along CB,

$$\therefore \quad \frac{100}{26} \text{ inches along } CA.$$

Hence the arm of the couple is $\frac{26}{3} - \frac{50}{13}$ inch;

therefore the moment of the couple is

$$3\left(\frac{26}{3} - \frac{50}{13}\right) \mathbf{F} \text{ by inch},$$

$$i.e., \quad 26 - 11{\cdot}5 \quad ,,$$

$$i.e., \quad 14{\cdot}5 \quad ,,$$

EXERCISE XXXVII.

1. Find the value of foot-ton of statical moment in terms of inch-pound.

2. A substance is weighed from either arm of an unequal balance, and its apparent weights are 9 lbs. and 4 lbs.; find the ratio of the arms.

3. An article placed in one pan of a balance weighs 425 grains, in the other 406 grains. What is its correct weight?

4. A body is weighed first in one pan and then in the other pan of a balance with unequal arms; the apparent weight of the body in the first experiment is

29·62 grammes, in the second it is 28·70 grammes. Find the true weight and the ratio of the arms.

5. If a shopkeeper use a false weight of $15\frac{7}{8}$ oz. for a pound, what will a true hundredweight appear to weigh?

6. A bar, 3 feet 4 inches long, rests horizontally over a peg, with two weights of $8\frac{1}{2}$ lbs. and $11\frac{3}{4}$ lbs. respectively hung at its ends. Find the distance of the peg from the smaller weight.

7. The arm of a steelyard is 30 inches long, the distance of the supporting hook from the hook for the weight is half an inch. What is the greatest mass which can be measured by means of a movable weight of 14 lbs.?

8. A steelyard takes a horizontal position when there is no body attached, and the hook is at a distance of 6 inches from the point of suspension; to balance a certain body a 14 lb. weight must be placed at 2·5 feet from the point of suspension. What is the mass of the body?

9. A well-balanced dogcart when loaded with 5 cwt. is found on a level road to exert a pressure of only 7 lbs. on the horse's back. If the distance between the pad and axle be 6 feet, find the position of the centre of gravity of the load.

10. A lever, centred at one end, is 15 feet long; a mass, hanging from the opposite end, is supported by an upward pressure of 28,270 lbs. at 13 feet from the fulcrum. Find the amount of the mass.

11. A valve, 3 inches in diameter, is held down by a lever and weight, the length of the lever being 10 inches, and the valve spindle being 3 inches from the fulcrum. Find the pressure per square inch which will lift the valve when the weight hung at the end of the lever is 25 lbs., disregarding the weight of the lever.

12. A safety valve having an area of 4 square inches is kept down by means of a lever, one end of which is hinged, and to the other a 14-lb. weight is attached. The lever is applied to the valve at one third of its length from the hinge. Find the maximum pressure per square inch which the valve will stand.

13. A mass of 3 lbs. is suspended from a lever at a distance of 2 inches from the fulcrum; at what point on the same side of the fulcrum must a force of 2 oz. act to balance the weight of the mass?

14. Two men are to carry a block of iron of 176 lbs. suspended from a uniform pole 14 feet long, each man's shoulder being 1 foot 6 inches from his own end of the pole; at what point of the pole must the block be suspended, in order that one of the men may bear 4/5 of the weight borne by the other?

15. Two parallel forces act at two points in a straight line, 6 inches apart, in opposite directions; their resultant is a force of 1 lb. acting at a point in the line 4 feet from the larger of the forces. Determine the amounts of the forces.

16. A lever AB of the first order, 8 feet long, has the fulcrum 2 feet from B; a weight of 5 lb. is hung from A, and one of 17 lbs. from B; putting the weight of the lever itself out of the question, from what point must a weight of 2·5 lbs. be hung to keep the lever horizontal?

17. Where, in the above case, must the centre of gravity be if we are justified in putting the weight of the lever out of the question?

18. A uniform bar 12 feet long is supported by two men, one of them at an end; where must the other be in order that he may support three fifths of the whole weight?

19. Three men are to carry a log which is of uniform size and density, and has a length of 12 feet. If one of the three lifts at an end, and the other two lift by means of a lever, how ought the lever to be applied in order that each man may bear one third of the weight?

20. A rod AB weighing 10 lbs. is found to balance about a point 8 feet distant from A; a weight of 6 lbs. is then fastened to A; about what point will the rod now balance?

21. A rod AB, whose length is 5 feet, and mass 10 lbs., is found to balance itself if supported on a fulcrum 3 feet from A; if this rod were placed horizontally on two points, one under A and the other under B, what pressure would it exert on each point?

22. A uniform bar 20 inches long and weighing 2 lbs. is used as a common steelyard, the fulcrum being 5 inches from one end. Find the greatest mass which can be weighed with a movable weight of 4 lbs.

23. Forces equal to the weights of 1 lb., 2 lbs., 3 lbs., 4 lbs. act along the sides of a square taken in order; find the magnitude and line of action of their resultant.

24. A uniform beam 18 feet long rests in equilibrium upon a fulcrum 2 feet from one end; having a weight of 5 lbs. at the farther and one of 110 lbs. at the nearer end to the fulcrum; find the weight of the beam.

SECTION XXXVIII.—GRAVITATION.

ART. 168.—**Law of Gravitation.** The law of gravitation, formulated by Newton, asserts that the attractive force between any two particles of matter is proportional to the mass of the attracted particle and to the mass of the attracting particle, and is inversely proportional to the square of the distance between them. Hence it is expressed by

$$k\ \mathsf{F} = \mathsf{M} \text{ attracted by } \mathsf{M} \text{ attracting per } (\mathsf{L} \text{ distance})^2.$$

The unit of force **F** might be defined by means of this law, by making $k = 1$; but when it is defined as in Art. 141, then k is not 1, unless by chance.

ART. 169.—**Intensity at Unit Distance.** When the attracting mass considered is constant, say m **M**, we deduce

$$mk \, \mathsf{F} = \mathsf{M} \text{ attracted per } (\mathsf{L} \text{ distance})^2.$$

This is called the intensity of the attraction at unit distance.

If, further, the distance considered is constant, say d **L**, we deduce

$$\frac{mk}{d^2} \mathsf{F} = \mathsf{M} \text{ attracted.}$$

This is called the intensity of the attraction at a given distance.

ART. 170.—**Intensity of a Field.** Round any distribution of matter there is a field of force, and at each point of the field there is a value for

F per **M** attracted.

The intensity at the place is the resultant of the several intensities due to the different elements of matter in the distribution.

A body of spherical form and of uniform density produces an intensity of force at any place outside it the same as if the whole of its mass were concentrated at the centre.

ART. 171.—**Potential.** When a particle is moved from one position to another position in a field of force, the amount of work done is independent of the path taken, depending only on the starting point and the final point. This gives us the idea of *change of potential*, which means the amount of work done in moving unit mass from one position to another. It is measured in terms of

W per **M** moved.

ELEMENTS OF THE SUN AND PLANETS.*

Name.	Mean Distance from the Sun.	Mean Diameter.	Density.
Sun, . .		860,000 mile	1·44 **M** per **V** = **M** per **V** of water.
Mercury, .	$35\frac{3}{4}$ (10^6 mile)	2,992 ,,	6·9 ,,
Venus, . .	$66\frac{3}{4}$,,	7,660 ,,	4·8 ,,
Earth, . .	$92\frac{1}{3}$,,	7,918 ,,	5·7 ,,
Mars, . .	141 ,,	4,211 ,,	4·2 ,,
Jupiter, . .	480 ,,	86,000 ,,	1·38 ,,
Saturn, . .	881 ,,	70,500 ,,	·75 ,,
Uranus, . .	1771 ,,	31,700 ,,	1·3 ,,
Neptune, .	2775 ,,	34,500 ,,	1·2 ,,

Name.	Period of Rotation.	Relative Weight at Surface.	Period of Revolution.
Sun, . .	25 to 26 days per turn.	27·71 **L** per **T** per **T** at surface = **L** per **T** per **T** at surface of earth.	
Mercury, .	24h. 5m. ,,	0·46 ,,	87·97 days per rev.
Venus, . .	23h. 21m. ,,	0·82 ,,	224·70 ,,
Earth, . .	23h.56m.4·09s. ,,	1·00 ,,	365·26 ,,
Mars, . .	24h.37m.22·7s. ,,	0·39 ,,	686·98 ,,
Jupiter, . .	9h. 55m. 20s. ,,	2·64 ,,	11·86 years per rev.
Saturn, . .	10h. 14m. ,,	1·18 ,,	29·46 ,,
Uranus, . .		0·90 ,,	84·02 ,,
Neptune, .		0·89 ,,	164·78 ,,

EXAMPLES.

Ex. 1. How far would a body fall towards the earth in one second from a point at a distance from the earth's surface equal to the earth's radius ?

The earth, being very approximately a sphere, attracts as if its mass were situated at its centre. Hence,

* Newcomb's *Popular Astronomy*, p. 542.

$$32 \text{ poundal} = \text{lb. per (radius)}^2,$$
$$2 \text{ radius},$$
$$\tfrac{32}{4} \text{ poundal per lb.};$$
$$\therefore \quad \tfrac{32}{4} \text{ ft. per sec. per sec.};$$
$$\therefore \quad \tfrac{32}{8} \text{ ft. per sec.}^2 \,;$$
$$1 \text{ sec.}$$
$$\therefore \quad \tfrac{32}{8} \text{ ft., } i.e., \ 4 \text{ ft.}$$

Ex. 2. The moon's mass is 136×10^{21} lbs., the moon's radius $5{\cdot}7 \times 10^6$ ft.; the mass of the earth is $11{,}920 \times 10^{21}$ lbs.; the radius of the earth 21×10^6 ft. Find how far a stone at the moon's surface would fall in a second, the disturbance due to attraction of the earth being neglected.

$$32 \text{ poundal per lb.} = 11{,}920 \times 10^{21} \text{ lb. attracting per } (21 \times 10^6 \text{ ft.})^2\,;$$
$$\therefore \quad \frac{32 \times 21^2 \times 10^{12}}{11{,}920 \times 10^{21}} \text{ poundal per lb.} = \text{lb. attg. per (ft. dist.)}^2,$$
$$136 \times 10^{21} \text{ lb. per } (5{\cdot}7 \times 10^6 \text{ ft.})^2\,;$$
$$\therefore \quad \frac{32 \times 21^2 \times 10^{12} \times 136 \times 10^{21}}{11{,}920 \times 10^{21} \times (5{\cdot}7)^2 \times 10^{12}} \text{ poundal per lb.},$$
$$i.e., \quad \frac{32 \times 21^2 \times 136}{11{,}920 \times (5{\cdot}7)^2} \text{ ft. per sec. per sec}\,;$$
$$\therefore \quad \frac{16 \times 21^2 \times 136}{11{,}920 \times (5{\cdot}7)^2} \text{ ft.} = \text{sec.}^2,$$
$$1 \text{ sec.}^2,$$
$$\therefore \quad 2{\cdot}5 \text{ ft.}$$

EXERCISE XXXVIII.

1. Suppose that the earth shrank until its diameter were 6,000 miles, what would be the effect on the weight of an inhabitant? The diameter of the earth is approximately 8,000 miles.

2. If we suppose the mass of the sun to be 300,000 times the mass of the earth, and its radius to be 100 times the radius of the earth, find the attraction at the surface of the sun of a mass which at the surface of the earth is attracted by the force one pound weight.

3. The diameter of Jupiter is 10 times that of the earth, and its mass 300 times that of the earth ; by how much per cent. of his former weight would the weight of a man be increased by being removed from the surface of the earth to the surface of Jupiter?

4. The intensity of gravity at the surface of the planet Jupiter being about 2·6 times as great as it is at the surface of the earth, find approximately the time which a heavy body would occupy in falling from a height of 167 feet to the surface of Jupiter.

5. Find the intensity of the earth's attraction at the distance of the moon, taking 32 feet per second per second as its value at the surface of the earth. The diameter of the moon's orbit is 480,000 miles.

CHAPTER FIFTH.

THERMAL.

SECTION XXXIX.—TEMPERATURE.

ART. 172.—**General Unit.** The temperature of a body is its thermal state considered with reference to its power of communicating heat to other bodies. Any unit of difference of temperature may be denoted by **Θ**.

ART. 173.—**Standard Difference and Derived Units.** For a standard difference of temperature Newton chose the difference between the thermal state of pure water when freezing and the thermal state of the same substance when boiling under the standard atmospheric pressure.

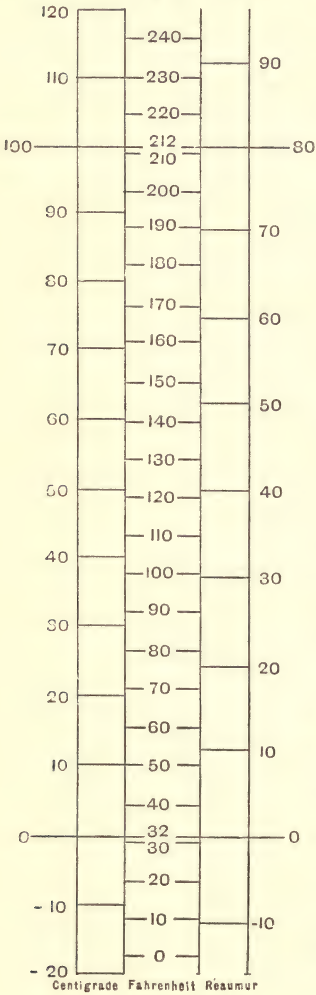
While there is unanimity in the choice of a standard, there is diversity in the mode of deriving a unit. The degree of Fahrenheit is the one hundred and eightieth part of the standard interval, while the degree of Celsius is the one hundredth part, and is called on that account the Centigrade degree. The degree of Réaumur is the eightieth part of the same interval. Hence the following equivalences,

9 degs. Fahr. = 5 degs. Cent. = 4 degs. Réaumur.

ART. 174.—**Scales of Temperature.** As temperature is an ordinal quantity it is necessary to fix upon an origin from which to reckon, especially as the state of no temperature cannot be

directly observed. Fahrenheit chose for origin the temperature which is 32 of his degrees below the freezing point of water; Celsius and Réaumur chose the lower fixed point, namely, the freezing point of water. In Fig. 24 the three scales are compared, the Fahrenheit directly with the two others.

Fig.24.



By the following rule a numerical comparative scale of the Centigrade and Fahrenheit degrees can be easily drawn out. On the Centigrade scale 0 is identical with 32 on the Fahrenheit; for every 5 added to the 0 add a 9 to the 32.

For the absolute zero of temperature see Art. 199.

Observe that the expression $n°$ F. is used to denote a temperature; while the expression *n deg. Fahr.* is used to denote a number of units of temperature. The former quantity is ordinal, the latter is not.

ART. 175.—**Use of Scales.** The Centigrade scale has been generally adopted where the metric system has been adopted. It is commonly used along with the C.G.S. system. The Fahrenheit scale is very generally used in English-speaking countries for purposes of ordinary life, and also for those of science, though the Centigrade scale is com-

ing into use among those who wish their results to be readily followed by foreigners. That of Réaumur is used to some extent on the Continent of Europe for medical and domestic purposes.

EXAMPLES.

Ex. 1. The point of maximum density of water is 4° Centigrade. What is the same point denoted by on the Fahrenheit scale?

First, to convert the 4 Cent. degrees into Fahr. degrees.

$$180 \text{ deg. Fahr.} = 100 \text{ deg. Cent.},$$

i.e., $9 \text{ deg. Fahr.} = 5 \text{ deg. Cent.},$

and $4 \text{ deg. Cent.};$

$\therefore$ $\frac{4 \times 9}{5}\text{deg. Fahr.},$

i.e., $7{\cdot}2 \text{ deg. Fahr.}$

Second, to find the ordinal number. The number is 7·2 degs. Fahr. above the freezing point, which is 32 degs. Fahr. above the zero of the Fahrenheit scale; therefore it is $7{\cdot}2 + 32°$ F., *i.e.*, $39°{\cdot}2$ F.

Ex. 2. Express on the Centigrade scale 85° F., 0° F., $-20°$ F.

We could proceed as above, first converting the degrees Fahrenheit into degrees Centigrade, and then subtracting the 32 degrees Fahrenheit previously reduced to degrees Centigrade. But it is more expeditious to subtract the 32 from the given number, and then convert the algebraic remainder, the result being the ordinal number required.

$$9 \text{ degs. Fahr.} = 5 \text{ degs. Cent.},$$

$$(85 - 32) \text{ degs. Fahr.};$$

$\therefore$ $\frac{(85-32) \times 5}{9}\text{degs. Cent.},$

i.e., $29°{\cdot}44$ C.

Similarly, $\frac{(0-32)5}{9}$ °C., *i.e.*, $-17°{\cdot}78$ C.;

and $\frac{(-20-32)5}{9}$ °C., *i.e.*, $-28°{\cdot}89$ C.

EXERCISE XXXIX.

1. Convert 212 degrees Fahrenheit into degrees Centigrade ; and 100 degrees Centigrade into degrees Fahrenheit.

2. Express 300° C. and 200° C. in terms of the Fahrenheit scale.

3. Express 500° F., 400° F., 300° F., 200° F., 100° F., in terms of the Centigrade scale.

4. Express the British standard temperature of 62° F. in terms of the Centigrade scale, and the French standard temperature of 3°·94 C. in terms of the Fahrenheit scale.

5. Express 327° F., 86° F., 70° F., and −30° F. in terms of the Centigrade scale.

6. Express on the Centigrade scale the following melting points :—

Lead, - - -	630° F.	Brass, - - -	1869° F.
Zinc, - - -	773° F.	Copper, - - -	1996° F.
Silver, - - -	873° F.	Cast-iron, - -	2786° F.

7. Express on the Fahrenheit scale the following boiling points :—

Alcohol, - -	78° C.	Mercury, - -	357° C.
Ether, - - -	35° C.	Sulphuric Acid, -	338° C.

8. The following melting points are expressed on the Centigrade scale, express them on the Fahrenheit scale :—

Mercury, - -	−39°·44.	Tin, - - -	235°.
Sodium, - -	90°.	Iron, - - -	1500°.

9. The point at which mercury freezes is indicated approximately by the same number on the Fahrenheit and Centigrade scales. Find the number.

10. At what point are the numbers on the scales of Fahrenheit and Réaumur identical ?

11. Find the first four coincidences above the freezing point of the Fahrenheit and Centigrade degree divisions.

12. What temperature is denoted on the Fahrenheit scale by double the number it is denoted by on the Centigrade scale ?

SECTION XL.—HEAT.

ART. 176.—**General Unit.**—Heat, being a species of energy, can be measured in terms of the unit of energy, for example, by the foot-pound, or the erg, or the joule (Art. 226). But for certain purposes it is found more convenient to define an independent thermal unit, which can afterwards be compared with the unit of energy. Any unit of heat is denoted by **H**.

ART. 177.—**Thermal Unit.** The thermal unit is commonly defined by the effect of heat in altering the temperature of a standard substance, when the whole of the heat communicated goes to produce alteration of temperature. The substance chosen is pure water about the temperature of its maximum density, that is, 4° C. The amount of heat required to be communicated to a mass of water to change its temperature from the m^{th} Θ to the n^{th} Θ is strictly proportional to the mass of the water, but it is not strictly proportional to the difference of temperature $n - m\ \Theta$. It depends upon the particular values of m and n, and is strictly proportional to the difference of temperature only when the difference is small. This is expressed by saying that the value k of the equivalence

$$k\,\mathsf{H} = \mathsf{M} \text{ of water by } \Theta \text{ difference}$$

depends upon the initial or average temperature of the change. The variation in the value of this equivalence is slight for the range between 0° C. and 4° C.; consequently the unit of heat may be defined by

$$1\,\mathsf{H} = \mathsf{M} \text{ of water by } \Theta \text{ difference};$$

on the understanding that for exact purposes the change is limited to the interval mentioned.

ART. 178.—**Special Thermal Units.** Different units of heat are obtained according to the special units of mass and of temperature adopted. Some of these have no special name, and in consequence are specified by the quantities on which they depend. We have the *pound of water by degree Fahrenheit*, which is the ordinary British unit; *the pound of water by degree Centigrade; the kilogramme of water by degree Centigrade*, which is the ordinary French unit; *the gramme of water by degree Centigrade.* The third of these units is denominated the *calorie*, and the fourth is denominated the *small calorie.*

ART. 179.—**Dynamical Equivalent of Heat.** By the mechanical, or, more appropriately, the dynamical equivalent of heat is meant

the value in terms of a unit of work of a thermal unit of heat. By his experiments Joule has found that

$$772{\cdot}55 \text{ foot-pounds} = \text{pound of water by degree Fahr.},$$

the intensity of gravity being that of the sea-level at Greenwich. For ordinary calculations the value 772 is sufficiently exact. The value for any other unit of work and any other unit of heat can be found by the arithmetical process of conversion. Thus

$$9 \text{ degs. Fahr.} = 5 \text{ degs. Cent.},$$

$$\therefore \quad 772 \times 9 \text{ foot-pounds} = 5 \text{ pound of water by deg. Cent.},$$

$$i.e., \quad 1{,}390 \text{ foot-pounds} = \text{pound of water by deg. Cent.}$$

In the case of the French units,

$$424 \text{ kilogrammetres} = \text{kgm. of water by deg. Cent.}$$

It is customary to denote by J any one of these three factors.

EXAMPLES.

Ex. 1. How much boiling water and how much freezing water must be taken to make up a pound of water at blood-heat, that is, 97° Fahr.?

Suppose x lb. of boiling water, then $1-x$ lb. of freezing water.

The change of temperature in the first case is $212-97$ deg. Fahr., therefore the heat given out is

$$x\,(212-97) \text{ lb. of water by deg. Fahr.}$$

In the second case the change of temperature is $97-32$ deg. Fahr., therefore the heat taken in is

$$(1-x)(97-32) \text{ lb. of water by deg. Fahr.}$$

Now, if there be no loss of heat to other bodies,

$$x(212-97) = (1-x)(97-32);$$

hence $$x = \tfrac{13}{36}, \text{ and } 1-x = \tfrac{23}{36}.$$

Answer—$\frac{13}{36}$ lb. of boiling, and $\frac{23}{36}$ lb. of freezing water.

Ex. 2. Reduce pound by deg. Cent. to ergs.

$$1{,}390 \text{ foot-pounds} = \text{lb. of water by deg. Cent.},$$

$$1{\cdot}356 \times 10^7 \text{ ergs} = \text{foot-pound};$$

$$\therefore 1{\cdot}356 \times 1{,}390 \times 10^7 \text{ ergs} = \text{lb. of water by deg. Cent.},$$

$$i.e., \; 1{\cdot}9 \times 10^{10} \text{ ergs} = \text{lb. of water by deg. Cent.}$$

Ex. 3. In one of Rumford's experiments, a horse working for 2 hours 30 minutes raised the temperature of a mass, equivalent in capacity for heat to 26·6 lbs. of water, by 180 degs. Fahr. Taking the work of one horse at 30,000 foot-pounds per minute, calculate from this experiment the dynamical equivalent of heat.

The heat produced was

$$26 \cdot 6 \times 180 \text{ lb. of water by deg. Fahr.}$$

Now 30,000 foot-pounds per minute,

150 minutes ;

$\therefore$ the work expended was $150 \times 30{,}000$ foot pounds.

$$\text{Hence } \frac{150 \times 30{,}000}{266 \times 18} \text{ foot-pounds} = \text{lb. of water by deg. Fahr.}$$

i.e., 940 foot-pounds = lb. of water by deg. Fahr.

EXERCISE XL.

1. Reduce pound of water by degree Fahrenheit to ergs.

2. Find the relation between the pound of water by degree Fahrenheit, and the kilogramme of water by degree Centigrade.

3. The temperature of a fluid is ascertained by the hand to be the same as that of a mixture of 3 pounds of water taken at 0° C. with 7 pounds of water taken at 100° C. ; what is the temperature of the fluid?

4. What is the amount of available heat in a quart bottle, filled with water at 95° C., and capable of being cooled to 10° C. ?

5. A mass of 100 lbs. falls 100 feet, and after striking a fixed obstacle rebounds 30 feet; calculate the value in terms of the British thermal unit of the work done.

6. A leaden bullet of 2 oz. strikes a target at a speed of 1,000 feet per sec., and is stopped. Find in terms of the British unit the whole amount of heat generated in the impact?

7. The Niagara Falls are 165 feet in height. Find by how much the temperature of the water will be increased by the fall, supposing that the whole energy of the water due to the fall is transformed into heat.

8. Given that the frictional resistance to a passenger train is 17 pounds per ton of load; what is the amount of work done against friction by a train of 75 tons in going a journey of 10 miles? Also what is the amount of heat developed?

9. By the consumption of a gramme of carbon 8,000 units of heat are produced ; if 40 per cent. of this be employed in raising a mass of one gramme, how high will it be raised, the intensity of gravity being 981 cm. per sec. per sec. ?

10. The combustion of one pound of coal raises the temperature of 100 gallons of water through 4·4 degrees Cent. What is the mechanical equivalent of the absolute thermal effect of the coal?

11. An engine consumes 40 lbs. of coal of such calorific power that the heat developed by the combustion of one lb. is capable of converting 16 lbs. of water at 100° C. into steam at the same temperature, and during the process the engine performs 16×10^7 foot-pounds of work. What percentage of the heat produced is wasted?

SECTION XLI.—THERMAL CAPACITY.

ART. 180.—**Thermal Capacity per Unit Mass.** The thermal capacity per unit mass of a substance, commonly called the thermal capacity, is the number of units of heat required to raise unit of mass of the substance one degree in temperature. It is expressed in the form

$$c \ \mathsf{H} = \mathsf{M} \text{ of substance by } \Theta \text{ difference.}$$

The value of c varies from temperature to temperature, increasing as the temperature gets higher.

When H is defined as

$$1 \ \mathsf{H} = \mathsf{M} \text{ of water by } \Theta \text{ difference,}$$

the thermal capacity of water is evidently 1, unless the temperature of the water differs much from the temperature at which the unit is defined.

When the mass is constant, say m M, then

$$mc \ \mathsf{H} = \Theta \text{ difference.}$$

This equivalence expresses the thermal capacity of a body.

When, on the other hand, the interval of temperature is constant, t_1 Θ to t_2 Θ, we have a rate of the form

$$k \ \mathsf{H} = \mathsf{M} \text{ of substance.}$$

When $t_2 - t_1$ is large, k is not equal to $(t_2 - t_1)c$, unless c is the average value of the thermal capacity for the interval.

ART. 181.—**Thermal Capacity per Unit Volume.** Given the thermal capacity per unit mass of a substance as

$$c\ \mathsf{H} = \mathsf{M} \text{ of substance by } \Theta \text{ difference,}$$

and the density of the substance as

$$\rho\ \mathsf{M} = \mathsf{V};$$

then by eliminating M we get

$$\rho c\ \mathsf{H} = \mathsf{V} \text{ of substance by } \Theta \text{ difference.}$$

This expresses what is called the thermal capacity per unit volume.

When the unit of heat is defined in terms of water, and the unit of mass by the density of water, the thermal capacity per unit volume for water is 1, unless the temperature of the water differs greatly from its standard temperature.

ART. 182.—**Relative Heat and Specific Heat.** Let the thermal capacities of two substances A and B be

$$c_1\ \mathsf{H} \text{ per } \mathsf{M} \text{ of } A = \Theta \text{ difference,}$$

and $$c_2\ \mathsf{H} \text{ per } \mathsf{M} \text{ of } B = \Theta \text{ difference;}$$

then $$c_1/c_2\ \mathsf{H} \text{ per } \mathsf{M} \text{ of } A = \mathsf{H} \text{ per } \mathsf{M} \text{ of } B.$$

This means that c_1 units of heat communicated to unit mass of the substance A are equivalent in respect of changing the temperature to c_2 units of heat communicated to unit mass of the substance B.

When the substance B is water, and H is a water-unit, the relative heat becomes the specific heat; c_2 becomes 1, and c_1 is equally the value of the thermal capacity per unit mass, and the specific heat referred to unit of mass.

The specific heat is expressed by

$$c_1\ \mathsf{M} \text{ of water by } \Theta \text{ difference per } \mathsf{M} \text{ of } A = \mathsf{M} \text{ of water by } \Theta \text{ difference per } \mathsf{M} \text{ of water;}$$

which is a ratio so far as dimensions of units are concerned.

It is evident that when the thermal capacity per unit volume is taken the relative heat will be

$$c_1\rho_1/c_2\rho_2\ \mathsf{H} \text{ per } \mathsf{V} \text{ of } A = \mathsf{H} \text{ per } \mathsf{V} \text{ of } B;$$

and that the specific heat referred to volume will be

$$c_1 s \text{ M of water by } \Theta \text{ difference per V of } A = \text{M of water by } \Theta \text{ difference per V of water,}$$

where s denotes the value of the specific mass of A.

ART. 183.—**Resistance to Change of Temperature; Water Equivalent.** Suppose that the thermal capacity of a substance A is

$$c_1 \text{ H} = \text{M of } A \text{ by } \Theta \text{ difference,}$$

and of a substance B

$$c_2 \text{ H} = \text{M of } B \text{ by } \Theta \text{ difference.}$$

By taking the reciprocals we get

$$1/c_1 \text{ M of } A = \text{H per } \Theta \text{ difference,}$$

and $$1/c_2 \text{ M of } B = \text{H per } \Theta \text{ difference;}$$

therefore c_2/c_1 M of A = M of B.

This expresses equivalent masses, the equivalence being in respect of resistance to change of temperature.

When B denotes water and H is defined in terms of water (Art. 177) c_2 is 1, and

$$c_1 \text{ M of water} = \text{M of } A.$$

This is the water equivalent of unit of mass of the substance.

When a body contains m_1 M of substance A, m_2 M of B, and m_3 M of C, then its water equivalent is

$$m_1 c_1 + m_2 c_2 + m_3 c_3 \text{ M of water.}$$

In a similar manner we derive

$$\rho_2 c_2/\rho_1 c_1 \text{ V of } A = \text{V of } B,$$

which expresses equivalent volumes with respect to the same property.

When A is water about its standard temperature

$$sc \text{ V of water} = \text{V of substance;}$$

this expresses water equivalent referred to volume.

Specific Heat Referred to Mass of Solids and Liquids.

s **H** per **M** of substance = **H** per **M** of water,

or s **M** of water = **M** of substance.

Pure substances in the solid state.	s	*Pure substances in the solid state—cont.*	s
Aluminium,	·207	Sulphur,	·18
Antimony,	·050	Tin,	·055
Bismuth,	·030	Zinc,	·093
Cadmium,	·055		
Copper,	·095	*Artificial substances in the solid state.*	
Gold,	·032		
Graphite,	·310	Brass,	·09
Ice,	·50	Glass,	·19
Iron,	·113	Steel,	·12
Lead,	·031		
Magnesium,	·245	*Pure substances in the liquid state.*	
Mercury,	·032		
Nickel,	·109	Alcohol,	·60
Platinum,	·033	Chloroform,	·23
Quartz,	·19	Mercury,	·033
Silver,	·056	Paraffin,	·68
Sodium,	·293	Turpentine,	·50

Art. 184.—**The Two Specific Heats of a Gas.** Let the specific heat, referred to mass, at constant pressure be

s_p **H** per **M** of gas at constant pressure = **H** per **M** of water about 4° C.;

and the specific heat, referred to mass, at constant volume be,

s_v **H** per **M** of gas at constant volume = **H** per **M** of water about 4° C.;

then we deduce

s_p/s_v **H** per **M** of gas at constant pressure = **H** per **M** at constant volume.

The value of s_p for a substance can be determined with comparative facility, but the determination of s_v is attended with

difficulty. The value of s_p/s_v can be determined from independent considerations, and is found to be about 1·405 for a perfect gas. Air, oxygen, nitrogen, and hydrogen, in their ordinary state approximate to the ideal perfect gas. In the following table the best experimental values of these three ratios are given.

The values of s_p and s_v are also the values of the thermal capacities, when **H** is a water-unit.

SPECIFIC HEAT OF A GAS AT CONSTANT PRESSURE, AND AT CONSTANT VOLUME; AND THE RATIO OF THE FIRST TO THE SECOND ACCORDING TO INDEPENDENT EXPERIMENTAL DETERMINATIONS.

s_p **H** per **M** of gas at constant pressure = **H** per **M** of water about 4° C.
s_v **H** per **M** of gas at constant volume = **H** per **M** of water about 4° C.
s_p/s_v **H** per **M** of gas at constant pressure = **H** per **M** of gas at constant vol.

Gas.	s_p.	s_v.	s_p/s_v.
Air, - - -	·238	·168	1·40
Oxygen, - -	·218	·155	
Nitrogen, - -	·244	·173	
Hydrogen, - -	3·41	2·41	1·39
Chlorine, - -	·12	·0928	1·32
Muriatic acid, -	·19	·130	1·39
Carbonic acid, -	·20	·172	1·30
Steam, - -	·480	·370	......
Marsh gas, - -	·593	·468	......
Olefiant gas - -	·388	·359	1·22
Chloroform, - -	·146	·140	......
Ammonia, - -	·52	·391	1·30
Ether, - - -	·46	·453	......

EXAMPLES.

Ex. 1.—Find the amount of heat which must be given to an iron armour plate 8 feet long, 7 feet wide, and 1 foot thick, in order to heat it from 12° C. to 900° C., assuming that its mean specific heat may be taken at ·2.

·2 lb. of water = lb. of iron,

480 lbs. of iron = cubic feet,

8×7 cubic feet;

$\therefore$ $\cdot 2 \times 480 \times 8 \times 7$ lb. of water,

and $900 - 12$ deg. Cent. difference;

$\therefore$ $\cdot 2 \times 480 \times 8 \times 7 \times 888$ lb. of water by deg. Cent.,

i.e., $4 \cdot 7 \times 10^6$ lb. of water by deg. Cent.

Ex. 2.—Ten grammes of sodic chloride at 91° having been immersed in 125 grammes of oil of turpentine at 13°, the temperature of the mixture was 16°. Find from these data the specific heat of sodic chloride, supposing no loss or gain of heat to have taken place from without, and taking the specific heat of oil of turpentine at 0·428.

$\cdot 428$ gm. water = gm. oil of turpentine,

125 gm. oil of turpentine;

$\therefore$ $125 \times \cdot 428$ gm. water,

and $16 - 13$ deg. Cent. rise;

$\therefore$ $3 \times 125 \times \cdot 428$ calories.

Suppose s gm. water = gm. sodic chloride,

10 gm. sodic chloride;

$\therefore$ $10\,s$ gm. water,

and $91 - 16$ deg. Cent. fall;

$\therefore$ $75 \times 10\,s$ calories.

As there is no loss or gain of heat from without,

$$3 \times 125 \times \cdot 428 = 75 \times 10\,s,$$

$$\therefore \quad s = \cdot 214.$$

Hence $\cdot 214$ gm. water = gm. sodic chloride.

Observe—When grammes are mentioned, the Centigrade scale is understood.

Ex. 3. Into a calorimeter at the temperature of 16° C. are thrown 8·5 gms. of water at a temperature of 84° C., and the temperature of the two becomes 18·5° C. What is the water-equivalent of the calorimeter?

The heat given out is

$$8 \cdot 5 \times (84 - 18 \cdot 5) \text{ gm. of water by deg. Cent.}$$

The change of temperature of the calorimeter is
$18{\cdot}5 - 16$ deg. Cent.; therefore the equivalent of its mass is

$$\frac{8{\cdot}5 \times (84 - 18{\cdot}5)}{18{\cdot}5 - 16} \text{ gm. of water,}$$

$$\textit{i.e.,} \qquad 222{\cdot}7 \text{ gm. of water.}$$

Ex. 4. A leaden bullet, weighing 2 ounces, strikes a target with a velocity of 1,400 feet per second, and is stopped. Find the whole heat generated in the impact. Find also the rise of temperature of the bullet, supposing that two-thirds of the heat generated goes to warm the bullet.

$$\tfrac{1}{2} \text{ foot-poundal} = \text{lb. by (ft. per sec.)}^2,$$

$\frac{1}{8}$ lb. by (1400 ft. per sec.)^2;

$$\therefore \qquad \frac{(1400)^2}{2 \times 8} \text{ foot-poundal,}$$

$$1 \text{ foot-pound} = 32{\cdot}2 \text{ foot-poundal,}$$

$$1 \text{ lb. of water by deg. Fahr.} = 772 \text{ foot-pound;}$$

$$\therefore \qquad \frac{(1400)^2}{2 \times 8 \times 32{\cdot}2 \times 772} \text{ lb. of water by deg. Fahr.,}$$

$$\textit{i.e.,} \qquad 4{\cdot}925 \text{ lb. of water by deg. Fahr.}$$

This is the whole heat generated in the impact.

The heat communicated to the bullet is

$$\tfrac{2}{3} \times 4{\cdot}925 \text{ lb. of water by deg. Fahr.,}$$

$$10{,}000 \text{ lb. of lead} = 314 \text{ lb. of water;}$$

$$\therefore \qquad \frac{2 \times 4{\cdot}925 \times 10{,}000}{3 \times 314} \text{ lb. of lead by deg. Fahr.}$$

But the bullet contains $\frac{1}{8}$ lb. of lead;
therefore its rise of temperature is

$$\frac{2 \times 4{\cdot}925 \times 10{,}000 \times 8}{3 \times 314} \text{ deg. Fahr.,}$$

$$\textit{i.e.,} \qquad 836 \text{ deg. Fahr.}$$

This is more than sufficient to raise it to its melting point.

EXERCISE XLI.

1. One pound of boiling water at 100° C. is placed in contact with one pound of mercury at 0° C., and heat is transmitted from the water to the mercury until both are at the same temperature. Calculate this temperature.

2. If 10 lbs. of mercury at the temperature of freezing water were poured into a vessel containing 3 lbs. of water at 50° C., how much would the water be cooled?

3. A ball of platinum whose mass is 200 grammes is removed from a furnace and immersed in 150 grammes of water at 0°. If we suppose the water to gain all the heat which the platinum loses, and if the temperature of this water rises to 30°, determine the temperature of the furnace.

4. Two lbs. of mercury at 100° C. are dropped into one half pound of water at 10° C. and stirred about with the water. Find the final temperature of the water and mercury.

5. A piece of iron of 60 ounces mass, and at the temperature of 100° C., is immersed in 120 ounces of water at 20° C. Find the resulting uniform temperature.

6. If the heat yielded by one kilogramme of water in cooling down from 100° C. to 0° C. were employed in heating ten kilogrammes of mercury initially at −20° C., to what temperature would the mercury be raised?

7. An iron saucepan weighing two kilogrammes contains 1·5 litres of water. Given that, when the vessel and its contents are heated from 0° to 80°, the water absorbs 6·58 times as much heat as the saucepan, determine the specific heat of the latter.

8. The specific heat of iron is ·113; how many pounds of iron at 250° C. must be introduced into an ice calorimeter in order to produce 2 pounds of water?

9. The mass of a copper calorimeter and stirrer is 135 grammes; 156·5 grammes of water at 15°·3 C. are put in; when 85 grammes of a metal at a temperature of 98° is dropped in, the resulting temperature is 19°·1. Find the specific heat of the metal.

10. Two hundred grammes of zinc are heated to the temperature of 99° C. and plunged into 200 grammes of water contained in an iron calorimeter having the temperature 14° C. Assuming that all the heat lost by the zinc is transferred to the water and calorimeter, and that the water equivalent of the calorimeter is 14 grammes; find the temperature to which the water rises.

11. The mass of a copper calorimeter is 110 grammes; 400 grammes of water at the temperature of 16° C. are put into the calorimeter; and then 60 grammes of a substance which has been heated to 98° C. are placed in the water whose temperature is now found to be 21° C. Find the specific heat of the substance.

12. 280 grammes of zinc are raised to the temperature of 97°, and immersed in 150 grammes of water at 14° contained in a copper calorimeter weighing 96

grammes. What will be the temperature of the mixture, supposing that there is no exchange of heat except among the substances mentioned? What is the water-equivalent of the calorimeter employed?

13. The mechanical equivalent of heat is 1,390 foot-pounds = pound of water by deg. Cent.; if the standard substance were iron instead of water, what would be the value of the equivalent?

14. How much will a mass of copper be raised in temperature by striking a hard non-conducting surface after a fall of 368 feet?

15. A bullet moving at 1,605 feet per second strikes an iron target; how much is its temperature raised?

16. With what velocity must a mass of iron strike a hard non-conducting substance to have its temperature raised by one degree Centigrade?

SECTION XLII.—LATENT HEAT.

ART. 185.—**Latent Heat.** The latent heat of a substance is the quantity of heat which must be communicated to unit mass of the substance in a given state in order to convert it into another state without any change of temperature. It is expressed in the form

$$l\ \mathsf{H}\ \begin{matrix}\text{absorbed}\\ \vdots \\ \text{given out}\end{matrix} = \mathsf{M}\ \text{changed}.$$

When the unit of heat is defined as in Art. 177, the above rate becomes

$$l\ \mathsf{M}\ \text{of water by}\ \Theta = \mathsf{M}\ \text{of substance}.$$

As the unit of mass appears in the same mode on either side of the equivalence, the value of l is independent of the unit of mass, depending only on the unit of temperature.

ART. 186.—**Changes of State.** There are two changes of state of the kind referred to, and each has a different name according to the direction of the change. A substance may change from the solid to the liquid state, or oppositely, from the liquid to the solid; and it may change from the liquid to the gaseous state, or oppositely, from the gaseous to the liquid. These changes are

denominated respectively, liquefaction or fusion, solidification, vaporization, condensation.

As pressure has only a slight effect on the change from liquid to solid, the latent heat of liquefaction or solidification is constant. But as pressure has a great effect on the change from liquid to gas. the latent heat of vaporization or condensation depends on the pressure at which the change takes place. If heat is absorbed in one change it is given out in the opposite change and according to the same rate.

ART. 187.—**Total Heat.** By total heat is meant the number of units of heat required to change unit of mass of a substance from one temperature to another temperature, the interval including a change of state. Thus for changing water-substance from water at 0° C. to steam at 100° C., and under a pressure of 760 mm.,

636 lb. of water by deg. Cent. = lb. of water changed.

LATENT HEAT OF FUSION, AND OF VAPORIZATION AT THE PRESSURE OF ONE ATMOSPHERE.

l **M** of water by deg. Cent. per **M** of substance changed.

Fusion.			Vaporization.		
Substance.	Melting-point in °C.	*l*.	Substance.	Boiling point in °C.	*l*.
Bismuth, - -	267	12·6	Alcohol, - -	78	202
Cadmium, - -	321	13·7	Ammonia, - -	12	294
Ice, - - -	0	79	Bisulphide of Carbon, - -	46	96
Iron, grey cast, -	1200	23	Bromine, - -	58	46
Lead, - - -	325	5·61	Carbonic Acid, -	4·6	48
Mercury, - -	−40		Ether, - -	35	90
Phosphorus, -	43	5·1	Mercury, - -	350	62
Platinum, - -	1779	27·2	Sulphur, - -	316	362
Silver, - -	999	21·1	Water, - -	100	536
Sulphur, - -	115	9·37			
Tin, - - -	230	13·8			
Zinc, - - -	415	28·1			

EXAMPLES.

Ex. 1. The latent heat of fusion of ice is 79·25 in terms of the degree Centigrade. Express this constant in terms of the degree Fahrenheit.

$$79{\cdot}25\ \mathsf{M} \text{ of water by deg. Cent.} = \mathsf{M} \text{ melted,}$$
$$\tfrac{9}{5} \text{ deg. Fahr.} = \text{deg. Cent.;}$$
$$\therefore\quad 79{\cdot}25 \times \tfrac{9}{5}\ \mathsf{M} \text{ of water by deg. Fahr.} = \mathsf{M} \text{ melted,}$$
$$\textit{i.e.,}\quad 142{\cdot}65\ \mathsf{M} \text{ of water by deg. Fahr.} = \mathsf{M} \text{ melted.}$$

Ex. 2. One pound of boiling water is poured over two pounds of powdered ice; what will be the temperature, and what the physical condition of the result?

Since the latent heat of water is

$$79 \text{ lb. of water by deg. Cent.} = \text{lb. of ice melted,}$$

it will require 158 of these units of heat to melt the 2 pounds of powdered ice. But the reduction of the one pound of boiling water to the temperature of 0° C. can give only 100 units of heat; hence a portion only of the ice will be melted. Suppose x lbs.; then

$$x \times 79 = 100,$$
$$x = 1{\cdot}27.$$

Hence the resulting temperature will be that of melting ice, and 1·27 lbs. of the ice will be melted.

Ex. 3. Assuming the latent heat of melting ice to be 142, that of steam at 212° to be 966, and the specific heat of ice to be 0·5; find the temperature of a solid weighing 40 lbs., and having a constant specific heat 0·1, which on being plunged into 1 lb. of snow at temperature zero would just convert it into steam at 212°, supposing that no heat is received from or imparted to any other bodies by the ice and the solid.

The total heat of the lb. of snow is

$$32 \times {\cdot}5 + 142 + 180 + 966 \text{ lb. of water by deg. Fahr.}$$

Suppose that the temperature of the solid is $t°$ F., then the heat it loses is

$$40 \times \cdot 1(t - 212) \text{ lb. of water by deg. Fahr.}$$

Hence
$$4\ (t - 212) = 1{,}304,$$
$$\therefore \quad t = 538.$$

Answer—538° Fahr.

Ex. 4. The latent heat of fusion of ice is 79·5, and its specific gravity is ·917. Ten grammes of metal at 100° C. are immersed in a mixture of ice and water, and the volume of the mixture is found to be reduced by 125 cubic millimetres without change of temperature. Find the specific heat of the metal.

$$s \text{ gm. of water} = \text{gm. of metal},$$
$$10 \text{ gm. of metal};$$
$$\therefore \quad 10s \text{ gm. of water},$$
$$\text{and } 100 \text{ deg. Cent. of fall};$$
$$\therefore \quad 100 \times 10s \text{ gm. of water by deg. Cent.}$$

$$\text{Again } \frac{917}{1000} \text{ gm. ice} = \text{gm. water};$$

$$\therefore \quad \frac{1000}{917} \text{ cc. ice} = \text{cc. water};$$

$$\therefore \quad \frac{1000 - 917}{917} \text{ cc. difference} = \text{cc. water},$$

$$\frac{125}{1000} \text{ cc. difference};$$

$$\therefore \quad \frac{125 \times 917}{1{,}000 \times 83} \text{ cc. water};$$

$$\therefore \quad \frac{125 \times 917}{1{,}000 \times 83} \text{ gm. water.}$$

Hence,
$$\frac{125 \times 917 \times 79 \cdot 5}{1{,}000 \times 83} \text{ gm. of water by deg. Cent.};$$

and, by equating the two quantities of heat,

$$s = \frac{125 \times 917 \times 79 \cdot 5}{1{,}000 \times 83 \times 1{,}000},$$
$$= \cdot 11.$$

EXERCISE XLII.

1. 1·6 lb. of ice at 0° C. mixed with 9 lb. of water at 20°, yielded 10·6 lb. of water at 5°; find the latent heat of water.

2. If a lb. of ice at 0° C. be dropped into 2 lbs. of water at 26°·5 C., how much of the ice will be melted?

3. How much ice at 0° C. can be converted into water at 0° C. by an ounce of steam at 100° C., if we assume heat to be transmitted from the steam only to the ice?

4. How much steam at 100° C. is required to raise the temperature of 54 ounces of water from 0° C. to 100° C.?

5. How many pounds of steam at 100° C. will just melt 20 pounds of ice at 0° C.?

6. It is found that a kilogramme of water at 100° C., mixed with a kilogramme of melting snow without loss of heat, gives two kilogrammes of water at the temperature of 10°·36; find the latent heat of water.

7. One lb. of steam at 100° C. is passed into a vessel containing 5 lbs. of water at the temperature of 20° C., and then condensed. What will be the temperature of the mixed steam and water?

8. Half a pound of powdered ice is mixed with 4 pounds of water at 8° C.; what is the result, and what the final temperature of the mixture?

9. It is found as the result of experiment that 25 grammes of copper at the temperature of 100° C. are just sufficient to melt 2·875 grammes of ice at 0°, so that water and copper are finally at 0°. Find from these data the specific heat of copper?

10. If a pound of steam at 100° C. be injected into a gallon of water at 15° C., calculate the temperature to which the water will be raised.

11. Into a mass of water at 0° C., 100 grammes of ice at − 12° are introduced; 7·2 grammes of the water freeze about the lump immersed, while its temperature rises to zero. What is the specific heat of the ice?

12. By proper arrangement a vessel, whose capacity for heat is to be neglected, containing 10 grammes of water, is reduced in temperature 15 degrees Cent. below the freezing point of water. A small spicule of ice is then dropped in; calculate the quantity of ice formed.

13. Two kilogrammes of steam at 100° are conducted into a copper vessel weighing $\frac{1}{2}$ kilogramme, and containing 30 kilogrammes of water at the temperature of 10°; find the temperature of the resulting mass after condensation, supposing no heat to be lost by radiation.

14. The heat produced by the complete combustion of one gramme of carbon in a calorimeter can convert 100 grammes of ice at 0° C. into water at 0° C. How many grammes of water could be raised by the same amount of heat from 0° C. to 1° C.?

SECTION XLIII.—EXPANSION OF SOLIDS AND LIQUIDS.

ART. 188.—**Rate of Linear Expansion.** The expansion of a substance in the form of a bar is proportional to the original length of the bar, and to the change of temperature, provided that the change of temperature is small. Hence the rate of expansion is expressed in the form

a L increment = L original by Θ difference,

or a L increment per L original = Θ difference,

or a L increment per L original per Θ difference.

It is evident from the expression of this rate that its value is independent of the magnitude of the unit of length, and depends only on the magnitude of the unit of temperature. It is commonly called the coefficient of expansion per degree Centigrade, or per degree Fahrenheit, as the case may be. Its value in the case of most substances varies slightly for different original temperatures.

This rate is similar in its nature to rate of interest (Art. 28), length taking the place of value, and temperature the place of time.

ART. 189.—**Coefficient of Expansion and Mean Rate of Expansion.** When the interval of temperature through which the solid is raised is constant, we have what is called the coefficient of expansion; it is expressed in terms of

k L increment per L original.

By the mean value of the rate of expansion for an interval is meant the coefficient of expansion for the interval divided by the number of units of temperature in the interval. When the interval is such as that between the freezing point and the boiling point of water, the mean value of the rate of expansion is very nearly the same as each of the true values.

MEAN COEFFICIENT PER DEGREE CENTIGRADE OF THE LINEAR EXPANSION OF SOLIDS.

Range from 0° C. to 100° C.

a L increment per L original per degree Cent.

METALLIC SOLID.	$a \times 10^4$.	NON-METALLIC SOLID.	$a \times 10^4$.
Elements.		*Elements.*	
Aluminium, ...	·234	Graphite, ...	·079
Bismuth, along axis, . .	·16	Sulphur, ...	·64
Bismuth, perpendicular to axis	·12	*Compounds.*	
Cadmium, ...	·31		
Copper, ...	·179	Brick, ...	·05
Gold, ...	·145	Glass, ...	·09
Iron, ...	·12	Granite, ...	·09
Lead, ...	·295	Sandstone, ...	·12
Magnesium, ...	·27	Slate, ...	·104
Platinum, ...	·09		
Silver, ...	·194	*Woods.*	
Tin, ...	·227	Range from 0° C. to 24° C.	
Zinc, ...	·29		
		Boxwood, across the fibre,	·61
Alloys or Compounds.		Boxwood, along the fibre, .	·03
		Oak, across the fibre, .	·54
Baily's Metal, ...	·177	Oak, along the fibre, . .	·05
Brass, 71 Cu + 29 Zn, .	·187	Pine, across the fibre, .	·34
Bronze, 86 Cu + 10 Sn + 4 Zn,	·180	Pine, along the fibre, .	·05
German Silver, ...	·18	Walnut, across the fibre, .	·48
Steel, ...	·111	Walnut, along the fibre, .	·07

ART. 190.—**Rate of Contraction.** Suppose that the mean rate of expansion for a particular solid between 0° C. and 100° C. is

a L increment per L original = deg. Cent. rise;

and that the solid experiences a change of t degs. Cent. within the interval and starting from 0° C. Then the rate of growth is

$1 + at$ L at $t°$ = L at 0°,

and the reciprocal of the rate of growth is

$$\frac{1}{1+at} \text{ L at } 0° = \text{L at } t°.$$

Now
$$\frac{1}{1+at} = 1 - at + (at)^2 - \text{etc.};$$

but as at in the case of any solid or liquid is very small compared with 1, the terms after the second may be neglected; so that $1 - at$ is practically equal to $1/(1 + at)$. Thus

$$\overline{1 - at}\ \mathsf{L} \text{ at } 0° = \mathsf{L} \text{ at } t°.$$

Hence

$$-\alpha\ \mathsf{L} \text{ increment per } \mathsf{L} \text{ original} = \text{deg. Cent. fall,}$$

or

$$\alpha\ \mathsf{L} \text{ decrement per } \mathsf{L} \text{ original} = \text{deg. Cent. fall;}$$

provided the original temperature is within the range from 0° to 100°, or not far beyond either extremity of the interval. The approximation is the same in principle as reckoning discount by banker's discount (Art. 30).

ART. 191.—**Cubical Expansion of Solids.** The term *dilatation* is sometimes used for cubical expansion. A solid may or may not have the same physical properties in all directions; in the former case it is said to be *isotropic*, in the latter *aeolotropic*. If the rate of expansion of an isotropic substance in any direction is

$$\alpha\ \mathsf{L} \text{ increment per } \mathsf{L} \text{ original} = \Theta \text{ rise,}$$

its rate of cubical expansion is (Art. 97)

$$3\alpha\ \mathsf{L}^3 \text{ increment per } \mathsf{L}^3 \text{ original} = \Theta \text{ rise.}$$

When a body is aeolotropic, it has three principal directions of expansion or contraction, which are at right angles to one another. If the values of the rate of expansion along these three directions are α, β, γ, the rate of cubical expansion is (Art. 97)

$$\overline{\alpha + \beta + \gamma}\ \mathsf{L}^3 \text{ increment per } \mathsf{L}^3 \text{ original} = \Theta \text{ rise.}$$

ART. 192.—**Expansion of Liquids.** The rate of expansion is expressed in the form

$$\alpha\ \mathsf{V} \text{ increment per } \mathsf{V} \text{ original} = \Theta \text{ rise.}$$

As the special values of α are small fractions, the same approximations may be used as in the case of solids.

When the interval of rise of temperature is constant we have a coefficient of expansion which is expressed in the form

$$k\ \mathsf{V} \text{ increment per } \mathsf{V} \text{ original;}$$

from which we derive

$$1 + k \text{ V final} = \text{V original.}$$

The dilatation or contraction which a substance experiences in passing from the solid to the liquid state is expressed in the same manner by a coefficient.

ART. 193.—**Coefficient of Expansion of Water.** Owing to the importance in Physics of water and of mercury, their change of volume has been studied with great care. Appended is a table of the best results in the case of water.

COEFFICIENT OF THE CHANGE OF VOLUME OF WATER.

$1 + k$ V at $t°$ C. = V at 4° C.; or $1/\rho$ cubic cm. = gm. at $t°$ C.

t	$1+k$ or $1/\rho$	t	$1+k$ or $1/\rho$	t	$1+k$ or $1/\rho$
0	1·00013	11	1·00035	50	1·0119
1	1·00007	12	1·00045	55	1·0144
2	1·00003	13	1·00057	60	1·0169
3	1·00001	14	1·00070	65	1·0197
4	1·00000	15	1·00084	70	1·0226
5	1·00001	20	1·00174	75	1·0257
6	1·00003	25	1·00287	80	1·0289
7	1·00007	30	1·00425	85	1·0322
8	1·00012	35	1·00586	90	1·0357
9	1·00018	40	1·00770	95	1·0394
10	1·00026	45	1·00972	100	1·0432

ART. 194.—**Apparent Dilatation of a Liquid.** By the *apparent* dilatation of a liquid is meant its dilatation as evidenced by the envelope in which it is contained. Let the apparent dilatation of the liquid be

$$m \text{ V increment per V original,}$$

and the dilatation of the envelope,

$$n \text{ V increment per V original;}$$

then $\quad 1 + m \text{ V apparent} = \text{V original},$

and $\quad 1 + n \text{ V true} = \text{V apparent};$

$$\therefore \quad (1 + m)(1 + n)\text{V true} = \text{V original.}$$

Since m and n are both small, the true dilatation of the liquid is

$$m+n \text{ V increment per V original.}$$

MEAN COEFFICIENT PER DEGREE CENT. OF THE CUBICAL EXPANSION OF LIQUIDS. Between 0° C. and 100° C.

a V increment per V original per degree Centigrade.

LIQUID.	$a \times 10^2$	LIQUID.	$a \times 10^2$
Alcohol, - - -	·126	Olive Oil, - - -	·08
Aniline, - - - -	·092	Petroleum, - - -	·104
Benzol, - - - -	·138	Sulphuric Acid, - -	·049
Bisulphide of Carbon, -	·147	Turpentine, - - -	·105
Bromine, - - -	·125	Water, pure, - - -	·043
Chloroform, - - -	·140	Water, ordinary sea, -	·05
Mercury, - - -	·018		

EXAMPLES.

Ex. 1. A piece of iron wire in a fence is 136 yards long in the middle of summer; how much shorter will it be in the middle of winter, in a climate where the maximum summer temperature is 88° F., and the minimum winter temperature $-5°$ F. The rate of expansion of iron per degree Centigrade is ·0000123.

·0000123 yard shorter per yard original = deg. Cent. fall,

$\frac{5}{9}$ deg. Cent. = deg. Fahr.;

∴ ·0000123 × $\frac{5}{9}$ yds. shorter per yd. original = deg. Fahr.,

i.e., ·0000068 yd. shorter = yd. original by deg. Fahr.

136 yd. original by 88 + 5 deg. Fahr.;

∴ 136 × 93 × ·0000068 yd. shorter,

i.e., ·086 yd. shorter,

or 3·1 inch shorter.

Ex. 2. An isotropic solid at 0° C. when immersed in water displaces 500 cubic inches, at 30° it displaces 503 cubic inches; find its mean coefficient of linear expansion between 0° and 30°.

$$503 \text{ cubic inch at } 30^\circ = 500 \text{ cubic inch at } 0^\circ;$$

$$\therefore \quad \frac{503-500}{500} \text{ cubic inch increment} = \text{cubic inch at } 0^\circ;$$

$$\therefore \quad \frac{503-500}{500 \times 30} \text{cubic inch increment per cubic inch at } 0^\circ = \text{deg.Cent.};$$

$$\therefore \quad \frac{1}{3}\,\frac{503-500}{500 \times 30} \text{ inch increment per inch at } 0^\circ = \text{deg. Cent.},$$

$$\textit{i.e.,} \quad \frac{1}{15000} \text{ inch increment per inch at } 0^\circ = \text{deg. Cent.}$$

Ex. 3. The linear contraction of a casting of tin is $\frac{1}{4}$ inch per foot; find the superficial contraction.

$$\tfrac{1}{4} \text{ inch} = \text{ft.};$$

$$\therefore \quad \tfrac{1}{48} \text{ ft.} = \text{ft.}$$

Hence

$$1 - \tfrac{1}{48} \text{ ft.} = \text{ft.};$$

$$\therefore \quad (1 - \tfrac{1}{48})^2 \text{ sq. ft.} = \text{sq. ft.};$$

$$\therefore \quad 1 - (1 - \tfrac{1}{48})^2 \text{ sq. ft. decrement} = \text{sq. ft.},$$

$$\textit{i.e.,} \quad \frac{1}{24} - \frac{1}{48^2} \text{ sq. ft. dect.} = \text{sq. ft. original},$$

$$\textit{i.e.,} \quad \frac{95}{2304} \text{ sq. ft. dect.} = \text{sq. ft. original.}$$

$\frac{1}{24}$ is a good approximation for the value arrived at.

Ex. 4. The true coefficient of expansion of mercury per degree Centigrade being 1/5550, and its apparent coefficient of expansion per degree Centigrade in glass being 1/6480; find the coefficient of cubical expansion of glass per degree Centigrade.

$$1 + \frac{1}{6480} \text{ V apparent} = \text{V original},$$

$$1 + x \text{ V true} = \text{V apparent};$$

$$\therefore \quad (1+x)\left(1 + \frac{1}{6480}\right)\text{V true} = \text{V original.}$$

But we are given $1 + \dfrac{1}{5550}$ V true = V original;

$$\text{hence} \quad 1+\frac{1}{5550}=(1+x)\left(1+\frac{1}{6480}\right);$$

$$\therefore \quad \frac{1}{5550}=x+\frac{1}{6480},$$

for each of the three quantities is small;

$$\therefore \quad x=\frac{1}{5550}-\frac{1}{6480},$$
$$=\cdot 0001802-\cdot 0001543,$$
$$=\cdot 0000259,$$
$$=\frac{1}{38700}.$$

Hence 1/38700 **V** increment per **V** original per degree Centigrade.

EXERCISE XLIII.

1. The coefficient of the linear contraction of a casting of tin is $\frac{1}{4}$ inch per foot; what is the coefficient of the volume contraction?

2. A lightning-rod made of copper measures 50 feet in length when at the temperature of 0° C.; find its length in summer when heated to the temperature of 27° C.

3. What effect will a rise of temperature of 25 degrees Centigrade have on a measuring chain, supposing that it is correct at 0° C.? What will be the effect of a fall of temperature of the same amount?

4. If a bar of iron has a length of 10 yards at 0° C., what will be its length at 30° C.?

5. Find how much an iron girder 100 feet long will expand between 32° F. and 68° F.

6. If the expansion of steel is two thirds that of brass under the same change of temperature, what will be the best arrangement of rods of these metals to form a gridiron pendulum?

7. Calculate the cooling effect of a cube of ice 2 feet in the side, taken at 0° C., and reaching 27° C. when its cooling power has been exhausted.

8. A solid weighs 320 grammes *in vacuo*, 240 grammes in distilled water at 4° C., and 242 grammes in water at 100° C., of which the density is 0·959 gms. per cc. Find the volume of the solid at these two temperatures, and deduce therefrom its mean coefficient of cubical expansion per degree Centigrade.

9. An isotropic solid when immersed in water at 10° C. displaces 600 cubic inches, and at 40° it displaces 604 cubic inches; find its mean rate of linear expansion between 10° and 40°.

10. A solid is weighed in a liquid at 0° C. and 100° C. The volume of the solid at 0° C. is unity, and at 100° C. 1·006. Also the loss of weight by weighing in the liquid is, at 0° C., 1,800 grains, and at 100° C. 1,750 grains. Find the coefficient of dilatation of the liquid.

11. In a vessel of glass of which the coefficient of expansion for the rise of temperature used is 1/7740, the apparent coefficient of expansion of mercury is 1/1296; find the true coefficient of expansion of mercury.

12. Suppose that an English barometer with a brass scale giving true inches at the temperature 62° F., reads 29·5 inches at 45° F.; what is the pressure in true inches of mercury reduced to the density it has at 32° F.?

13. Find the reading of a thermometer, the bulb of which is plunged in water at the temperature of 100° C., while the stem is exposed to air at the temperature of 10° C.

SECTION XLIV.—EXPANSION OF GASES.

ART. 195.—**Rate of Expansion.** When a gas receives an increase of temperature, it may either increase in volume or it may increase in pressure. Hence the rate of expansion due to increase of temperature is specified under the condition that the pressure remains constant; and the rate of increase of pressure due to increase of temperature under the condition that the volume remains constant. The former rate is expressed in the form

$$\alpha_p \; \mathsf{V} \text{ increment per } \mathsf{V} \text{ at standard temperature} = \Theta \text{ rise.}$$

The standard temperature chosen is the freezing point of water. The value α_p remains constant whatever the change of temperature from the standard temperature, provided that the gas is not brought near its point of condensation.

For air we have,

$$1/273 \; \mathsf{V} \text{ increment per } \mathsf{V} \text{ at } 0^\circ \text{ C.} = \text{deg. Cent. rise.}$$

It will be observed that the values for other gases not easily condensed are very approximately the same.

ART. 196.—**Rate of Increase of Pressure.** The rate of increase of pressure under constant volume is expressed in the form

$$\alpha_v \; \mathsf{P} \text{ increment per } \mathsf{P} \text{ original} = \Theta \text{ rise;}$$

where P denotes any unit of pressure per square inch.

The value of the rate of increase of pressure under constant volume is theoretically the same; thus

1/273 **P** increment per **P** at 0° C. = deg. Cent. rise.

COEFFICIENT PER DEGREE CENT. OF THE EXPANSION OF A GAS; AND THE COEFFICIENT PER DEGREE CENT. OF THE CHANGE OF PRESSURE OF A GAS.

a_p **V increment per V original per degree Cent.;**
a_v **P increment per P original per degree Cent.**

Gas.	$a_p \times 10^2$	$a_v \times 10^2$
Air, - - - -	·367	·366
Oxygen, - - -		·367
Nitrogen, - - -	·367	·367
Hydrogen, - - -	·366	·366
Carbonic acid, - -	·371	·368
Sulphurous acid, - -	·391	·385
Steam, - - -	·415	

ART. 197.—**Derived Rates.** The rate of expansion for a gas is similar in its nature to rate of simple interest, consequently the derived rates are similar.

Since for a gas

$$\frac{1}{273}\ \mathbf{V} \text{ increment per } \mathbf{V} \text{ at } 0^\circ \text{ C.} = \text{deg. Cent. rise;}$$

therefore for a change to t° C.

$$\frac{t}{273}\ \mathbf{V} \text{ increment} = \mathbf{V} \text{ at } 0^\circ,$$

and

$$1 + \frac{t}{273}\ \mathbf{V} \text{ at } t^\circ = \mathbf{V} \text{ at } 0^\circ,$$

and, reciprocally,

$$\frac{1}{1 + \frac{t}{273}}\ \mathbf{V} \text{ at } 0^\circ = \mathbf{V} \text{ at } t^\circ.$$

Hence $$1 - \frac{1}{1 + \frac{t}{273}} \mathbf{V} \text{ decrement per } \mathbf{V} \text{ at } t^\circ = \text{deg. Cent. fall,}$$

i.e., $$\frac{t}{273}/1 + \frac{t}{273} \mathbf{V} \text{ dect. per } \mathbf{V} \text{ at } t^\circ = \text{deg. Cent. fall,}$$

i.e., $$\frac{t}{273} - \left(\frac{t}{273}\right)^2 + \mathbf{V} \text{ dect. per } \mathbf{V} \text{ at } t^\circ = \text{deg. Cent. fall.}$$

Hence the value $t/273$ is correct only when $t/273$ is a small fraction. The constant changes its value according to the initial temperature selected.

The above rate applies to any change from 0° to a lower temperature, provided the substance is not brought near its point of condensation. It is modified to

$$\frac{1}{273} \mathbf{V} \text{ decrement per } \mathbf{V} \text{ at } 0^\circ = \text{deg. Cent. fall.}$$

Art. 198.—**Change from a Temperature other than the Standard Temperature.** To find the volume of a mass of gas originally at t_1° C. when changed to t_2°, the pressure being constant.

It is done in two steps, by supposing that the gas is reduced from t_1° to 0° and then raised from 0° to t_2°.

$$\frac{1}{1 + \frac{t_1}{273}} \mathbf{V} \text{ at } 0^\circ = \mathbf{V} \text{ at } t_1^\circ,$$

$$1 + \frac{t_2}{273} \mathbf{V} \text{ at } t_2^\circ = \mathbf{V} \text{ at } 0^\circ\,;$$

$$\therefore \quad \frac{1 + \frac{t_2}{273}}{1 + \frac{t_1}{273}} \mathbf{V} \text{ at } t_1^\circ = \mathbf{V} \text{ at } t_2^\circ,$$

i.e., $$\frac{273 + t_2}{273 + t_1} \mathbf{V} \text{ at } t_2^\circ = \mathbf{V} \text{ at } t_1^\circ.$$

Art. 199.—**Absolute Zero of Temperature.** If temperature be reckoned, not from the freezing point of water, but from a point

273 degrees Centigrade lower, then the volume of a constant mass of gas will always be proportional to its temperature, provided that the pressure is maintained constant throughout. Hence the connection will be expressed by

$$m \vee \text{volume} = \Theta \text{ temperature.}$$

The temperature 273 degrees Centigrade below the freezing point of water is called the absolute zero of temperature. It means the temperature at which the pressure or the volume of a mass of gas would vanish, on the supposition that the same rate of expansion held throughout which holds for the gaseous state.

Examples.

Ex. 1. 500 cubic centimetres of oxygen gas are measured when the temperature is 20° C., and the temperature is then raised to 40° C., the pressure meanwhile remaining constant. What is the volume of the oxygen at the latter temperature? The coefficient of the expansion of oxygen per degree Centigrade is $\frac{11}{3000}$.

$$500 \text{ cc. at } 20^\circ,$$

$$\frac{1}{1 + \frac{20 \times 11}{3000}} \text{cc. at } 0^\circ = \text{cc. at } 20^\circ,$$

$$1 + \frac{40 \times 11}{3000} \text{cc. at } 40^\circ = \text{cc. at } 0^\circ;$$

$$\therefore \quad 500 \frac{1 + \frac{40 \times 11}{3000}}{1 + \frac{20 \times 11}{3000}} \text{cc. at } 40^\circ,$$

$$\textit{i.e.,} \quad 500 \frac{3000 + 440}{3000 + 220} \text{cc. at } 40^\circ,$$

$$\textit{i.e.,} \quad 534 \text{ cc. at } 40^\circ.$$

Ex. 2. Find the mass of 1,000 cubic centimetres of dry air at 80° C. and the pressure of 25 cm. of mercury. 1,000 cubic centi-

metres of dry air at 0° C. and 76 cm. pressure have a mass of 1·293 grammes.

By Art. 134,

$$1{\cdot}293 \text{ gm. per } 1000 \text{ cc. at } 0^\circ = 76 \text{ cm. pressure,}$$

$$25 \text{ cm. pressure;}$$

$$\therefore \quad \tfrac{25}{76}\, 1{\cdot}293 \text{ gm.} = 1000 \text{ cc. at } 0^\circ,$$

$$1 \text{ cc. at } 0^\circ = 1 + \tfrac{80}{273} \text{ cc. at } 80^\circ,$$

$$1000 \text{ cc. at } 80^\circ;$$

$$\therefore \quad \tfrac{25}{76}\, 1{\cdot}293 \times \frac{273}{273+80} \text{gms.,}$$

$$\textit{i.e.,} \quad {\cdot}329 \text{ gms.}$$

EXERCISE XLIV.

1. A given mass of air occupies a volume of 600 cubic inches at the temperature of 20° C.; find the volume which the air will occupy at 100° C., supposing the pressure to remain constant.

2. A mass of gas occupying a volume of 273 cubic inches at 0° C. is raised in temperature to 150° C. If it be allowed to expand under constant pressure during the process, what will be its new volume?

3. One hundred cubic centimetres of air at 0° C. are heated to 300° C. under constant pressure. What will be the volume of the air at the higher temperature?

4. A thousand cubic inches of air at the temperature of 30° C. are cooled down to zero, and at the same time the external pressure upon the air is doubled. What is its volume reduced to?

5. Find the temperature to which 500 cubic centimetres of air, measured at 15° C. must be raised in order that the volume of the air may become 700 cubic centimetres, no change of pressure taking place meanwhile.

6. Twenty litres of air are taken at 16° C. and 74 cm. pressure; find the volume of the air at 0° C. and 76 cm. pressure.

7. One thousand cubic inches of gas are taken when the barometer stands at 30·5 inches, and the temperature is 16° C. Find the volume of this gas when the pressure is 29·5 inches and the temperature 12°.

8. Find the absolute zero on the Fahrenheit and on the Réaumur scale.

9. A substance, of the approximate specific gravity 3·2, weighs 180 grammes in dry air of 730 reduced mm. pressure and temperature of 16° C. Also the approximate specific gravity of the weights against which it is weighed is 8·5. Find the real weight of the substance, assuming that the weight of one litre of dry air at 0° C. and 760 reduced millimetres pressure is 1·293187 grammes.

SECTION XLV.—THERMAL CONDUCTIVITY.

ART. 200.—**Conductivity.** By the thermal conductivity of a substance is meant the rate connecting the current of heat with the gradient of temperature, when there is a steady flow of heat through the substance. It is expressed in the form

$$k\ \mathsf{H} \text{ per } \mathsf{T} \text{ per } \mathsf{S} \text{ cross-section} = \Theta \text{ per } \mathsf{L} \text{ normal.}$$

By "L normal" is meant unit of length along the line of flow, and "Θ per L normal" expresses what is called the *gradient of temperature*, after the analogy of gradient of gravity (Art. 72).

The reciprocal idea is *thermal resistance ;* it is expressed by

$$1/k\ \Theta \text{ per } \mathsf{L} \text{ normal} = \mathsf{H} \text{ per } \mathsf{T} \text{ per } \mathsf{S} \text{ cross-section.}$$

When the unit of heat is a dynamical unit, we have conductivity expressed in terms of

$$\mathsf{W} \text{ per } \mathsf{T} \text{ per } \mathsf{S} = \Theta \text{ per } \mathsf{L}.$$

For example, by

erg per sec. per sq. cm. = deg. Cent. per cm.;

or, which is the same thing, by

erg per sec. per sq. cm. per (deg. Cent. per cm.).

When the unit of heat is a thermal unit, we have

$$\mathsf{M} \text{ of water by } \Theta \text{ per } \mathsf{T} \text{ per } \mathsf{S} = \Theta \text{ per } \mathsf{L}\,;$$

for example,

gm. of water by deg. Cent. per sec. per sq. cm. = deg. Cent. per cm.

A value, expressed in terms of this kind of unit, is independent of the magnitude of Θ; for Θ enters to the same power in the two members of the equivalence. When the units are allowed to cancel one another as much as possible, there remains M/TL, which expresses the dimensions of the unit.

ART. 201.—**Thermometric Conductivity.** Suppose that the conductivity of a substance is

$$k\ \mathsf{M} \text{ of water by } \Theta \text{ per } \mathsf{T} \text{ per } \mathsf{S} = \Theta \text{ per } \mathsf{L},$$

and that the density of water is

$$\rho\ \mathsf{M} = \mathsf{V}\,;$$

then, by substitution,

$$\frac{k}{\rho}\,\mathsf{V} \text{ of water by } \Theta \text{ per } \mathsf{T} \text{ per } \mathsf{S} = \Theta \text{ per } \mathsf{L}.$$

Suppose further that the specific heat of the substance, referred to volume, is

$$s\,\mathsf{V} \text{ of water} = \mathsf{V} \text{ of substance};$$

then, by a second substitution,

$$\frac{k}{s\rho}\,\mathsf{V} \text{ of substance by } \Theta \text{ per } \mathsf{T} \text{ per } \mathsf{S} = \Theta \text{ per } \mathsf{L}.$$

The idea here expressed is called by Clerk-Maxwell the thermometric conductivity of a substance.*

Its dimensions, if a systematic unit, are

$$\mathsf{L}^2/\mathsf{T}.$$

THERMAL CONDUCTIVITY.

k (gm. of water by deg. Cent.) per sec. per cm.^{2} per (deg. Cent. per cm.).
Range from 0° C. to 100° C.

METALLIC SUBSTANCE.		NON-METALLIC SUBSTANCE.	
Elementary.	k	*Solid.*	$k \times 10^2$
Aluminium,	·35	Glass,	·05
Bismuth,	·015	Ice,	·5
Cadmium,	·21	Marble,	·52
Copper,	·80	Sandstone,	·53
Iron,	·17	Slate,	·47
Iron, wrought,	·18	*Liquid.*	
Lead,	·078	Water,	·14
Magnesium,	·38	*Gaseous.*	$k \times 10^3$
Mercury,	·017	Air,	·054
Silver,	1·09	Hydrogen,	·34
Tin,	·15	Nitrogen,	·052
Zinc,	·29	Oxygen,	·053
Compound.		Carbonic acid,	·038
Brass,	·25	Ammonia,	·06
German silver,	·081	Marsh gas,	·06
Steel,	·14		

* *Heat*, p. 235.

ART. 202.—**Relative Conductivity and Resistance.** Let the conductivity of two substances A and B be

$$k_1 \text{ H per T per S} = \Theta \text{ per L},$$

and $$k_2 \text{ H per T per S} = \Theta \text{ per L}.$$

Then $$1/k_1 \text{ S per L of } A = \text{H per T per } \Theta,$$

and $$1/k_2 \text{ S per L of } B = \text{H per T per } \Theta;$$

therefore $$k_1/k_2 \text{ S per L of } B = \text{S per L of } A.$$

This is the form for expressing the relative resistance in terms of the cross-section and length of the conductor.

The reciprocal is

$$k_2/k_1 \text{ L per S of } B = \text{L per S of } A;$$

and it expresses the relative conductivity.

EXAMPLES.

Ex. 1. How much heat is transmitted per day per square metre of surface, across a slab of rock 10 centimetres thick, whose sides differ in temperature by half a degree Centigrade, supposing the conductivity of the rock to be ·004? The centimetre is the unit of length, and the unit of heat is the quantity of heat required to raise the temperature of one gramme of water one degree.

·004 gm. of water by deg. Cent. per sec. per sq. cm.

= deg. Cent. per cm.,

$$\frac{1}{2 \times 10} \text{ deg. Cent. per cm.};$$

$$\therefore \quad \frac{\cdot 004}{20} \text{ gm. of water by deg. Cent.} = \text{sec. by sq. cm.},$$

$$60 \times 60 \times 24 \text{ sec.} = \text{day},$$

$$100^2 \text{ sq. cm.} = \text{sq. metre};$$

$$\therefore \quad \frac{\cdot 004 \times 60^2 \times 24 \times 100^2}{20} \text{ gm. of water by deg. Cent.}$$

= day by sq. metre,

i.e., $1{\cdot}728 \times 10^5$ gm. of water by deg. Cent. per day per sq. metre.

Ex. 2. Calculate the conductivity in the following instance. A square metre of a substance, one centimetre thick, has one side kept at 100° C., and the other by means of ice at 0° C.; and in the course of 30 minutes one kilogramme of ice is melted by this operation.

We shall express conductivity in terms of

Calories per min. per sq. cm. = deg. Cent. per cm.

79 calories = kgm. of ice melted,

1 kgm. of ice melted = 30 minutes ;

$\therefore$ $\frac{79}{30}$ calories per minute.

Hence

$\frac{79}{30}$ calories per min. per 100^2 sq. cm. = 100 deg. Cent. per cm.,

$\therefore$ $\frac{79}{30 \times 100^3}$ calories per min. per sq. cm. = deg. Cent. per cm.

i.e., $2{\cdot}6 \times 10^{-6}$ calories per min. per sq. cm. = deg. Cent. per cm.

EXERCISE XLV.

1. How many gramme-degrees of heat will be conducted in an hour through an iron bar two square centimetres in section and four centimetres long, its two extremities being kept at the respective temperatures of 100° C., and 178° C., the mean conductivity of iron being ·12. The units are a gramme, a centimetre, a second, and a degree centigrade?

2. Calculate the quantity of heat lost per hour from each square metre of the surface of an iron steam-boiler 0·8 centimetres in thickness, when the temperature of the inner surface of the boiler is 120°, and that of the outer surface $119\frac{1}{2}$°, the coefficient of conductivity of iron being 11·5, referred to the centimetre as the unit of length, the minute as the unit of time, and the quantity of heat required to raise the temperature of a gramme of water from 0° C. to 1° C. as the unit of heat.

3. Find how much heat is conducted in an hour across a plate of copper, one square metre in area, 16 centimetres thick, one side of the plate being kept 25 degs. Cent. hotter than the other. The conductivity of copper for heat is 1·108 in centimetre-gramme-second units.

4. The thermal conductivity of iron is about 0·0133, the units being the foot, minute, and degree Centigrade. Find how much heat per hour is lost by a

boiler of $\frac{1}{4}$ inch plate, whose surface is 10 square yards, and which contains water at 110° C., the external surface of the boiler being kept at 100° C.

5. Find how much heat is conducted in half an hour through an iron plate 2 centimetres thick and 1000 square centimetres in area, the temperature of the two sides being kept at 0° C. and 20° C.

6. The inside of the wall of a house is kept at 16° C., while the outside is at 0° C.; the wall is of solid stone, and two feet thick. How much heat is lost across it per square foot per hour? Take the conductivity at 0·001 lb. of water by deg. Cent. per sec. per sq. ft. = deg. per ft. thick.

7. The conductivity of silver in terms of the millimetre, second, deg. Cent., and water is 109; express it in terms of the C.G.S. thermal unit.

8. From the table deduce the thermal resistance relatively to silver, of lead, copper, mercury, German silver.

CHAPTER SIXTH.

ELECTRICAL.

SECTION XLVI.—MAGNETIC.

ART. 203.—**Unit Magnetic Pole.** A quantity of magnetism concentrated at a point is called a magnetic pole. The repulsive or attractive force exerted by one magnetic pole on another is proportional to the amount of magnetism of the one pole and to the amount of magnetism of the other pole, and is inversely proportional to the square of the distance between the poles. The law is the same as that of gravitation (Art. 168), with the exception that there may be repulsion as well as attraction. Let P denote any unit of magnetism, concentrated at a point; then the law is expressed by

$$k\ \mathsf{F} = \mathsf{P} \text{ repelled by } \mathsf{P} \text{ repelling per } (\mathsf{L} \text{ distance})^2;$$

or, using the symbols for "by" and "per" (Art. 79),

$$k\ \mathsf{F} = \mathsf{P} \text{ repelled} \times \mathsf{P} \text{ repelling } / \ (\mathsf{L} \text{ distance})^2.$$

By means of this law we are enabled to define P in terms of L, M, T; for F is already so defined (Art. 79). Take the case when the poles are of equal strength, then

$$k\ \mathsf{F} = \mathsf{P}^2 \ / \ (\mathsf{L} \text{ distance})^2;$$

$$\therefore\ \ 1/k\ \mathsf{P}^2 = \mathsf{F} \times (\mathsf{L} \text{ distance})^2,$$

or

$$1/\sqrt{k}\ \mathsf{P} = \sqrt{\mathsf{F}} \times \mathsf{L} \text{ distance}.$$

Thus P can be defined in magnitude by assigning a special value to k, and the most convenient value is 1. Hence

$$1\ \mathsf{P}^2 = \mathsf{F} \times \mathsf{L}^2,$$

$$= \mathsf{M} \times \mathsf{L}/\mathsf{T}/\mathsf{T} \times \mathsf{L}^2.$$

It is evident from the expression for the unit that the dimensions of P are $\frac{1}{2}$ for M, $\frac{3}{2}$ for L, and -1 for T.

It is assumed that the medium is air.

ART. 204.—**Special Units.** In the C.G.S. system F becomes the dyne, and L the centimetre. Hence

$$\mathsf{P}^2_{\text{c.g.s.}} = \text{dyne by centimetre}^2.$$

Before the establishment of the C.G.S. system, it was customary in Britain to use for a unit pole that obtained by taking F as the grain by ft. per sec. per sec., and L as the foot. In that case

$$\mathsf{P}^2_{\text{f.g.s.}} = 1/7000 \text{ poundal by ft.}^2.$$

It was proposed by Clausius at the Paris Congress of Electricians (Art. 221) to give the name of *weber* to the unit magnetic pole of the practical system of units. The denomination was not adopted on account of its having been used to denote a unit of current. It would be defined by

$$1 \text{ weber} = 10^8 \sqrt{\text{dyne}} \text{ by centimetre}.$$

ART. 205.—**Intensity of Magnetic Field.** We have then the law,

$$1\,\mathsf{F} = \mathsf{P} \text{ repelled} \times \mathsf{P} \text{ repelling} \,/\, (\mathsf{L} \text{ distance})^2.$$

If the strength of the repelling pole is $m\,\mathsf{P}$, then

$$m\,\mathsf{F} = \mathsf{P} \text{ repelled} \,/\, (\mathsf{L} \text{ distance})^2; \qquad (1)$$

and this expresses the *intensity at unit distance* of the magnetic field round the attracting pole.

If, further, the distance is constant, say $d\,\mathsf{L}$, then the *intensity at this distance* is

$$\frac{m}{d^2}\,\mathsf{F} = \mathsf{P} \text{ repelled}. \qquad (2)$$

Suppose that the repulsion is due not to one pole, but, as is always the case, to a distribution of magnetism. In the neighbourhood of any distribution of magnetism, or of a current flowing in a conductor, there is a field of magnetic force, just as in

the neighbourhood of the earth there is a field of gravitational force. At each point of the magnetic field there is a certain value of

$$\mathsf{F} \text{ per } \mathsf{P} \text{ repelled,} \tag{3}$$

which is the resultant of the intensity at that point of all the elements of which the distribution is made up.

The C.G.S. unit is dyne per $\mathsf{P}_{\text{c.g.s.}}$

Sir W. Thomson has suggested the denomination of *gauss* for the practical unit of intensity of field.

ART. 206.—**Magnetic Potential.** To move a magnetic pole from one position to another in a magnetic field involves an amount of work which is independent of the path taken, and which is proportional to the amount of magnetism in the pole moved, provided that the amount of magnetism of the pole is not large enough to change sensibly the intensity of the field. It must be a particle of magnetism compared with the magnetism producing the field. This gives us the idea of *magnetic potential*, which is expressed in terms of

$$\mathsf{W} \text{ per } \mathsf{P} \text{ repelled.}$$

In the C.G.S. system it is expressed by

$$\text{erg per } \mathsf{P}_{\text{c.g.s.}} \text{ repelled.}$$

As the idea involves two points, or rather two surfaces, we consider either difference of magnetic potential; or, if we consider magnetic potential simply, we imply that the zero surface is at an infinite distance; that is, a surface at every point of which the intensity of the field is inappreciable.

ART. 207.—**Magnetic Moment.** When a magnet has the form of a long thin bar, there are two equal and opposite poles situated near the extremities. Such a magnet, when placed in a field of magnetic force, with its axis transverse to the direction of the force at the place, experiences a couple. Let the strength of a

pole of the magnet be m **P**, its length l **L**, and the intensity of the field at the place i **F** per **P**; then the couple experienced is

iml **F** by **L** arm.

The dependence of this couple, so far as it depends on the magnet itself, is expressed by the factor

ml **P** by **L** distance between poles.

This is the idea of *magnetic moment.*

ART. 208.—**Intensity of Magnetization.** Consider a bar magnet l **L** long and of uniform cross-section a **L**^{2}, which has been magnetized longitudinally, so that its poles are near its extremities. Suppose this bar divided longitudinally and cross-wise into small bars. The magnetic moment of such small bar is proportional to its length and to its cross-section. Hence we have the idea of *intensity of magnetization,* which is expressed in the form

i **P** by **L** = **L** long by **L**^{2} cross-section.

In the case of the bar magnet considered, i is constant throughout both in direction (the direction of the axis of the moment) and in amount; hence the moment of the bar is

ila **P** by **L**.

When a magnet is not uniformly magnetized, the direction and magnitude of the intensity of magnetization vary from point to point; and in consequence one value of i can be considered constant only within a small region.

ART. 209.—**Declination, Dip, Horizontal Intensity.** The earth acts as a magnetic body in producing a field of magnetic force in its neighbourhood. The intensity at any point in the field is fully specified by stating its magnitude and the direction of the lines of force. It is, however, more convenient to measure the horizontal component of the intensity than the intensity itself. The direction of the line of force is specified by the *declination*, that is, the angle between the North and the direction of a magnetic needle free to move in a horizontal plane (Fig. 25); and by

the *dip* or *inclination*, that is, the angle between the former direction and the direction assumed by a magnetic needle free to move in that vertical plane (Fig. 26).

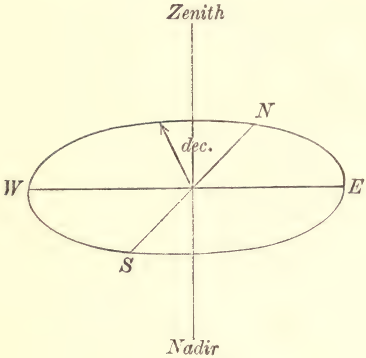


Fig. 25

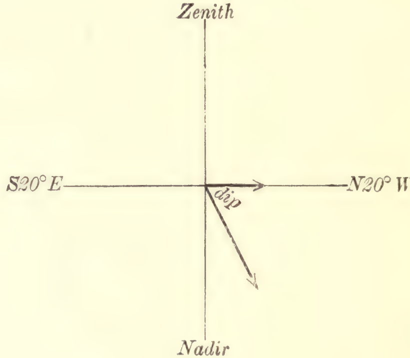


Fig. 26

Let the horizontal component be h **F** horizontal per **P**, and the dip δ degrees; then the total intensity is

$$h \sec \delta \ \mathbf{F} \text{ along per } \mathbf{P}.$$

MAGNETIC ELEMENTS OF TOWNS IN GREAT BRITAIN, FOR JAN., 1884.*

	Declination.	Dip.	Total Intensity in dynes per $\mathbf{P}_{c.g.s.}$
Greenwich, . . .	18° 10′ *W*	67° 30′	·472
Bristol, . . .	19° 30′ *W*	67° 45′	·474
Manchester, . .	20° 0′ *W*	68° 50′	·478
Dublin, . . .	22° 15′ *W*	69° 30′	·481
Newcastle, . . .	19° 55′ *W*	69° 45′	·480
Edinburgh, . . .	21° 10′ *W*	70° 30′	·484

* Lupton's *Numerical Tables*, p. 54.

ART. 210.—**Determination of the Horizontal Intensity.** When a magnet performs small oscillations freely in a horizontal plane, under the influence of the magnetic field of the earth only, the following relation holds true

$$4\,\pi^2(\mathsf{M} \times \mathsf{L}^2) \times (\text{vibration}/\mathsf{T})^2 = (\mathsf{F}/\mathsf{P}) \times (\mathsf{P} \times \mathsf{L})\,;$$

in which M by L^2 is the unit for moment of inertia about the centre of figure (Art. 161), F per P the unit for the horizontal intensity, and P by L the unit for the magnetic moment of the magnet.

EXAMPLES.

Ex. 1. Two magnetic poles, 7 and 9 C.G.S. units respectively, are placed at a distance of 5 cm. apart; find the force in grammes between them.

$$1 \text{ dyne} = \mathsf{P}_{\text{c.g.s.}} \times \mathsf{P}_{\text{c.g.s.}} \,/\, (\text{cm. distance})^2,$$

$$7\ \mathsf{P}_{\text{c.g.s.}} \times 9\ \mathsf{P}_{\text{c.g.s.}} \,/\, (5 \text{ cm.})^2;$$

$$\therefore \qquad \frac{7 \times 9}{25} \text{ dyne}\,;$$

$$\therefore \qquad \frac{7 \times 9}{25 \times 981} \text{ gramme-weight},$$

i.e., $\cdot 0026$ gramme-weight.

Ex. 2. A magnetic needle, the magnetic moment of which remains constant, is suspended so as to move freely in a horizontal plane. When deflected from the magnetic meridian at three different places on the earth's surface, it is observed to oscillate 7·5, 8·3, 10·4 times respectively in one minute. Compare the intensities of the earth's horizontal magnetic force at the three places.

As the moment of inertia and the magnetic moment of the needle are constant, we have (Art. 161)

$$k\,\mathsf{F} \text{ horizontal per } \mathsf{P} = (\text{vibration per minute})^2.$$

Hence the three horizontal intensities are as

$$(7{\cdot}5)^2 \quad : \quad (8{\cdot}3)^2 \quad : \quad (10{\cdot}4)^2\,;$$

i.e., $100 \quad : \quad 122 \quad : \quad 192.$

Ex. 3. The absolute horizontal intensity of the earth's magnetic force at London is 3·8 in terms of the foot-grain-second unit. Find its value in terms of the C.G.S. unit.

The intensity at a place is expressed in terms of **F** per **P**. Now

$$1\ \mathsf{F} = \mathsf{M} \times \mathsf{L}/\mathsf{T}/\mathsf{T},$$

and $$1\ \mathsf{P}^2 = \mathsf{M} \times \mathsf{L}/\mathsf{T}/\mathsf{T} \times \mathsf{L}^2;$$

therefore the change-factor for **F** per **P** is $\dfrac{\sqrt{m}}{t\sqrt{l}}$.

Now $$\frac{1}{15\cdot4}\ \text{gramme} = \text{grain},$$

and $$\frac{12}{\cdot394}\ \text{cm.} = \text{ft.};$$

$$\therefore \qquad 3\cdot8\sqrt{\frac{\cdot394}{15\cdot4 \times 12}}\ \text{dyne per } \mathsf{P}_{\text{c.g.s.}}$$

$$\begin{array}{llll}
\log \cdot394 = & \bar{1}\cdot59550 & \qquad \log 15\cdot4 = & 1\cdot18752 \\
 & 2\cdot26670 & \qquad \log 12 \quad = & 1\cdot07918 \\
\hline
 & 2)\bar{3}\cdot32880 & & 2\cdot26670 \\
 & \bar{2}\cdot66440 & & \\
\log 3\cdot8 = & 0\cdot57978 & & \\
\hline
 & \bar{1}\cdot24418 & &
\end{array}$$

Ans.—·175 dyne per $\mathsf{P}_{\text{c.g.s.}}$.

EXERCISE XLVI.

1. If the horizontal intensity of the earth's magnetic force at a place is 3·8, and the vertical 8·5, what is the total intensity?

2. When two long magnets of equal strength are placed in a straight line at a distance of 1 mm., the repulsive force is 3,290 dynes. Find the strength of a pole.

3. A small freely suspended magnet performs 72 oscillations per minute under the action of another magnet. How many would it perform if the distance between them were half as great again?

4. At various places on the earth's surface a declination needle vibrates 70, 60, and 50 times per minute. Compare the horizontal intensity at the three places.

5. Find the number of foot-grain-second units of intensity of magnetic field equivalent to one C.G.S. unit.

6. The horizontal component of terrestrial magnetic force at Glasgow is about ·16 C.G.S. unit. Find to two places its amount in terms of the foot-grain-second unit.

7. The dip at Paris in 1880 was 66°, and the total intensity ·47 dynes per $P_{c.g.s.}$ What was the horizontal intensity?

8. If the value of the magnetic moment of a magnet be 10, when the units of length, mass, and time are the centimetre, gramme, and second, what will be the value of the magnetic moment when the units are the metre, kilogramme, and minute?

9. The bearing of a ship's compass from a station on shore is *N.* 44° 20′ *E.*, and the bearing of the station by the ship's compass, taken at the same time, is *S.* 50° 50′ *W.* What is the deviation for this position of the ship's head?

10. Find by means of Art. 161 the moment of inertia about its centre of figure of a bar magnet, which is 5 cm. long, and has a section of 2 mm. square.

11. The horizontal intensity of the earth's magnetic field at Göttingen was found by Gauss to be 1·782 millimetre-milligramme-second units. Express it in terms of the C.G.S. unit.

SECTION XLVII.—ELECTROSTATIC.

ART. 211.—**General Unit of Electricity.** Any unit of quantity of electricity may be denoted by **Q**.

There are two laws, by means of either of which the unit of electric quantity may be defined; namely, the electrostatic law, and the electromagnetic law. The electrostatic law states how the repulsive or attractive force between two quantities of electricity depends on the magnitude of each, and on the distance between them. The electromagnetic law states how the intensity of the magnetic field depends on the current in the influencing wire. A unit of quantity defined by the former law, as well as any derived unit, is called an electrostatic unit; while a unit of quantity, defined by the latter law, as well as any derived unit, is called an electromagnetic unit.

ART. 212.—**Electrostatic Unit of Quantity.** The law of re-

pulsion or attraction for two quantities of electricity condensed on two small spheres is precisely similar to that for two magnetic poles. The repulsive force exerted by the one quantity on the other is proportional to the magnitude of the one and to the magnitude of the other, and is inversely proportional to the square of the distance between the centres of the spheres. Let Q denote any unit of electric quantity (Art. 211), then the law is stated by

$$k\ \mathsf{F} = \mathsf{Q} \text{ repelled} \times \mathsf{Q} \text{ repelling}/(\mathsf{L} \text{ distance})^2.$$

When the two quantities of electricity are equal, we have

$$k\ \mathsf{F} = \mathsf{Q}^2/(\mathsf{L} \text{ distance})^2;$$

$$\therefore \quad 1/k\ \mathsf{Q}^2 = \mathsf{F} \times (\mathsf{L} \text{ distance})^2,$$

or

$$1/\sqrt{k}\ \mathsf{Q} = \sqrt{\mathsf{F}} \times \mathsf{L} \text{ distance}.$$

The systematic unit is defined by the condition

$$1\ \mathsf{Q}^2 = \mathsf{F} \times (\mathsf{L} \text{ distance})^2.$$

In the case of the C.G.S. system F is the dyne, and L is the centimetre; so that

$$1 \text{ electrostatic } \mathsf{Q}^2_{\text{c.g.s.}} = \text{dyne by (cm. distance)}^2.$$

It is implied that the attraction is across air.

ART. 213.—**Intensity; Potential.** The ideas of intensity of electric force at unit distance, and of intensity at a given point, are precisely similar to those for magnetic force, and are derived in a similar manner.

We have also the corresponding idea of electric potential, the general unit for which is

$$\mathsf{W} \text{ per } \mathsf{Q} \text{ repelled.}$$

It is supposed that the quantity of electricity repelled is so small as not to alter sensibly the field of electric force when it changes its position.

For zero surface of potential the surface of the earth is generally chosen.

ART. 214.—**Electric Density.** A charge of electricity, when at rest on a conducting body, resides entirely on the surface. The

density of such a distribution is a surface density, and is accordingly expressed in the form

$$\sigma \ \mathsf{Q} \text{ per } \mathsf{S}.$$

ART. 215.—**Electric Capacity.** When a quantity of electricity is at rest on a conductor, the value of the potential, that is, of **W** per **Q** repelled, at any point of the surface of the conductor is the same. The quantity of electricity on the conductor is proportional to that potential, provided that all conductors in the neighbourhood are in connection with the earth. Hence the idea of the *capacity of a conductor*, which is expressed in the form

$$c \ \mathsf{Q} \text{ charge per } (\mathsf{W} \text{ per } \mathsf{Q} \text{ repelled}),$$

or

$$c \ \mathsf{Q} \text{ charge} = \mathsf{W} \text{ per } \mathsf{Q} \text{ repelled}.$$

This idea is analogous to that of the capacity of a body for heat (Art. 180), electricity taking the place of heat, and potential the place of temperature.

ART. 216.—**The Capacity of Different Forms of Conductors.** The value c depends on the form and size of the conductor, but not on the nature of its material, provided it is conducting.

For a sphere of conducting material, having a radius r **L**, and at a practically infinite distance from all other conductors, the capacity is $r \ \mathsf{Q} \text{ charge} = \mathsf{W} \text{ per } \mathsf{Q}$ repelled.

For example, a sphere of 5 cm. radius has a capacity of

$$5 \ \mathsf{Q}_{\text{c.g.s.}} = \text{erg. per } \mathsf{Q}_{\text{c.g.s.}} \text{ repelled}.$$

For a spherical Leyden jar formed of two nearly equal concentric conducting surfaces of radius r **L**, and separated by a thickness of air d **L**, the capacity is

$$\frac{r^2}{d} \mathsf{Q} \text{ charge} = \mathsf{W} \text{ per } \mathsf{Q} \text{ repelled};$$

or

$$\frac{s}{4\pi d} \mathsf{Q} \text{ charge} = \mathsf{W} \text{ per } \mathsf{Q} \text{ repelled},$$

where s L^2 is the area of one of the surfaces. This formula multiplied by k (Art. 217) holds for the ordinary form of Leyden jar.

For a condenser formed of two equal parallel plates separated

by a thickness d **L** of air, the capacity per unit of area of a plate is

$$\frac{1}{4\pi d}\ \mathsf{Q} \text{ charge per } \mathsf{L}^2 = \mathsf{W} \text{ per } \mathsf{Q} \text{ repelled.}$$

ART. 217.—**Specific Inductive Capacity.** Let the capacity of a condenser, such as that mentioned in Art. 216, when air is the insulating substance, be

$$c_1\ \mathsf{Q} \text{ charge, for Air} = \mathsf{W} \text{ per } \mathsf{Q}\ ;$$

and when paraffin is the insulating substance

$$c_2\ \mathsf{Q} \text{ charge, for Paraffin} = \mathsf{W} \text{ per } \mathsf{Q}.$$

It was first shown by Faraday that c_1 and c_2 are not necessarily equal. From these two capacities we derive

$$c_2/c_1\ \mathsf{Q} \text{ charge, for Paraffin} = \mathsf{Q} \text{ charge, for Air.}$$

This rate expresses the *relative inductive capacity.* As air is the reference substance, it also expresses the *specific inductive capacity*, c_2/c_1 being commonly denoted by k.

The capacity of a condenser formed of an insulating substance other than air, is k times that of an equal condenser having air for the insulating substance.

In the following table the gases are compared with vacuum, and the values given are those which have been determined by Boltzmann :—

SPECIFIC INDUCTIVE CAPACITY.

k_a **Q** charge, for substance = **Q** charge, for air.
k_o **Q** charge, for substance = **Q** charge, for vacuum.

INSULATING SUBSTANCE.	k	INSULATING SUBSTANCE.	k
Solid.	k_a	*Liquid.*	k_a
Beeswax,	1·9	Benzene,	1·5
Caoutchouc, pure,	2·3	Oil of turpentine,	1·5
Caoutchouc, vulcanized,	2·9	Petroleum,	1·4
Ebonite,	2·7		
Glass, flint,	6·8	*Gas.*	k_o
Glass, plate,	6·1	Air,	1·00059
Guttapercha,	4·2	Carbonic acid,	1·00095
Mica,	5	Carbonic oxide,	1·00069
Paraffin,	2·1	Hydrogen,	1·00026
Resin,	2·6	Marsh gas,	1·00095
Shellac,	3·3	Nitrous oxide,	1·00099
Sulphur,	3·4	Olefiant gas,	1·00131

ART. 218.—**Energy of a Charge.** Suppose that a conductor, having a capacity of c **Q** charge = **W** per **Q** moved, has been charged with q **Q**. What is the energy of this charge?

Suppose that the q **Q** has been moved up to the conductor in n equal portions from such a distance that the repulsive force of the charge placed on the conductor was, at that distance, inappreciable. There is no charge on the conductor to repel the first n^{th} portion on its being moved up. The rate of the work which must be done when the second portion is brought up, is

$$\frac{q}{nc}\ \mathbf{W} \text{ per } \mathbf{Q} \text{ moved};$$

hence the work done is $\dfrac{q}{nc} \times \dfrac{q}{n}\mathbf{W}$.

The potential in the third case is

$$\frac{2q}{nc}\ \mathbf{W} \text{ per } \mathbf{Q} \text{ moved, and the work } \frac{2q}{nc} \times \frac{q}{n}\ \mathbf{W}.$$

The potential in the n^{th} or last case is

$$\frac{(n-1)q}{nc}\ \mathbf{W} \text{ per } \mathbf{Q} \text{ moved; and the work is } \frac{(n-1)q}{nc} \times \frac{q}{n}\ \mathbf{W}.$$

Hence the total work is, since the potential increases uniformly,

$$\frac{n}{2}\left\{\frac{0+(n-1)q}{nc}\right\}\frac{q}{n}\mathbf{W}.$$

$$i.e., \qquad \frac{n(n-1)}{2}\left(\frac{q}{n}\right)^2\frac{1}{c}\mathbf{W}.$$

When n is a very large number, $n-1$ is equivalent to n, and the above expression becomes

$$\tfrac{1}{2}\frac{q^2}{c}\mathbf{W}.$$

Compare Art. 117.

EXAMPLES.

Ex. 1. Two equal small spheres, charged with quantities of electricity represented by the numbers 2 and 4, attract each other with a force represented by 10, when the distance between them

is 5. If the spheres are allowed to touch, and then separated by a distance 8, what force will they exert on one another?

Let the unit symbols F, Q, and L denote the arbitrary units mentioned.

$$10\,\mathsf{F} = 2\,\mathsf{Q} \text{ by } 4\,\mathsf{Q} \text{ per } (5\,\mathsf{L})^2;$$

$$\therefore \quad \frac{10 \times 25}{2 \times 4}\,\mathsf{F} = \mathsf{Q} \text{ by } \mathsf{Q} \text{ per } \mathsf{L}^2.$$

Since the spheres have equal radii, they have equal capacities, and therefore equal charges after separation. Hence

$$3\,\mathsf{Q} \text{ by } 3\,\mathsf{Q} \text{ per } (8\,\mathsf{L})^2;$$

$$\therefore \quad \frac{10 \times 25 \times 9}{2 \times 4 \times 64}\,\mathsf{F},$$

$$\textit{i.e.,} \quad 4{\cdot}4\ \mathsf{F}.$$

Ex. 2. Two small spheres of radii 3 mm. and 4 mm. respectively are in electrical connection, at the potential of 500 C.G.S. units, and at the distance of 6 cm. apart; find the force in grammes tending to separate them.

The capacity of the former sphere is

$$\cdot 3\,\mathsf{Q}_{\text{c.g.s.}} = \text{erg per } \mathsf{Q}_{\text{c.g.s.}},$$

$$500 \text{ erg per } \mathsf{Q}_{\text{c.g.s.}};$$

$$\therefore \quad 150\,\mathsf{Q}_{\text{c.g.s.}}$$

Similarly the charge on the latter sphere is $200\,\mathsf{Q}_{\text{c.g.s.}}$

$$\text{Now } 1 \text{ dyne} = \mathsf{Q}_{\text{c.g.s}} \text{ by } \mathsf{Q}_{\text{c.g.s.}} \text{ per (cm. dist.)}^2,$$

$$\frac{150 \times 200}{36} \quad \text{"} \qquad \text{"}$$

$$\therefore \quad \frac{150 \times 200}{36} \text{ dyne};$$

$$\therefore \quad \frac{150 \times 200}{36 \times 981} \text{ gramme-weight},$$

$$\textit{i.e.,}\ \cdot 85 \text{ gm.-wt.}$$

Ex. 3. Of two Leyden jars of the same kind of glass, the coatings of one measure each 1 sq. ft., and the glass is $\frac{1}{10}$ inch thick; the coatings of the other measure each 3 sq. ft., and the glass is

$\frac{1}{5}$ inch thick. The knobs of both are placed at the same time in contact with the prime conductor of an electrical machine, so that on working the machine they are both charged. Show what are the relative charges of the jars, and the relative amounts of heat produced by discharging them.

The capacity of a Leyden jar is

$$\frac{ks}{4\pi d}\ \mathbf{Q}\ \text{charge} = \mathbf{W}\ \text{per}\ \mathbf{Q}\ \text{repelled.}$$

When two jars of the same substance are at the same potential, we derive

$$\frac{s}{d}\ \mathbf{Q}\ \text{charge in former} = \frac{s'}{d'}\mathbf{Q}\ \text{charge in latter.}$$

Hence in the case mentioned,

$$10\ \mathbf{Q}\ \text{in former} = 3 \times 5\ \mathbf{Q}\ \text{in latter,}$$

$$\textit{i.e.,}\quad 2 \quad ,, \quad = 3 \quad ,,$$

Again, the amount of energy in a jar is $\frac{1}{2}qv$ **W**; hence the relative amounts of energy and therefore of heat developed in discharging is

$$2\ \mathbf{W}\ \text{in former} = 3\ \mathbf{W}\ \text{in latter.}$$

EXERCISE XLVII.

1. Find in terms of the C.G.S. unit the quantity of electricity which will attract an equal quantity at the distance of a metre with a force of 100 dynes.

2. Two insulated spheres, whose diameters are 5 and 8 centimetres, are charged with equal quantities of positive electricity; determine their relative potentials.

3. Two spheres of 5 cm. and 10 cm. diameter are charged with 25 and 30 units of electricity respectively; they are then connected by a long thin wire, and separated. What will now be the respective charges on the spheres?

4. If a globe one metre in diameter be insulated and charged to a potential of 7 electrostatic units; what is the amount of the charge?

5. One pole of a powerful battery is connected to earth, and a long insulated wire projects from the other end. Two insulated metal balls, of 1 inch and 5 inch diameter respectively, are put one after the other in contact with the end of the insulated projecting wire. What are the comparative quantities and densities of the electricities on the two balls?

6. A spherical conductor of 5 cm. diameter has a charge of 6 electrostatic units; what is the density of the distribution?

7. Equal quantities of electricity are placed on spheres of 1 centimetre and 1 decimetre diameter; compare the densities of the distributions.

8. The charge on a sphere of 4 inches diameter is allowed by means of a long thin wire to distribute itself over another sphere of 6 inches diameter. Compare the energy of the final with that of the original distribution.

9. The areas of the armatures of three condensers, exactly alike in all other respects, are as the numbers 3, 4, 5. Find their relative charges when at the same potential.

10. Five units of electricity are conducted into the interior of a Leyden jar of 200 sq. cm. surface, and 6 units of electricity are conducted into the interior of a similar jar of 300 sq. cm. surface. Compare the heat developed by discharging each.

11. A Leyden jar is charged from an electric machine, an unit jar being interposed, and ten discharges of the unit jar occur. Compare the energy expended by the person working the machine in each successive time of charging the unit jar.

12. A condenser is formed of two concentric spheres, one of which is 100 mm. and the other 101 mm. in radius. The specific inductive capacity of the dielectric is 2. The condenser is charged with 1,000,000 electrostatic units. Calculate in calories the amount of heat developed by the discharge.

SECTION XLVIII.—ELECTROMAGNETIC.

ART. 219.—**Electromagnetic Unit of Quantity.** As before, let Q denote any unit of quantity of electricity; then the strength of a uniform current of electricity flowing round a wire will be expressed in terms of Q per T. When a uniform current flows round a circular arc, the intensity of the magnetic field produced at the centre of the arc is directly proportional to the strength of the current, and to the length of the circular arc, and inversely proportional to the square of the radius of the arc. Thus we have the law

$$k\,\mathsf{F}/\mathsf{P} = (\mathsf{Q}/\mathsf{T}) \times \mathsf{L}\text{ arc}/(\mathsf{L}\text{ radius})^2\,;$$

a reciprocal form of which is

$$1/k\,\mathsf{Q}/\mathsf{T} = (\mathsf{F}/\mathsf{P}) \times (\mathsf{L}\text{ radius})^2/\mathsf{L}\text{ arc}.$$

It has been already shown that the units F and P can be defined systematically in terms of L, M, T. Hence Q is defined syste-

matically by making $k = 1$. It is implied that the medium is air.

ART. 220.—**Ratio of the two Units of Quantity.** Since both are units for the same thing, there is an equivalence

$$n \text{ electrostatic } \mathbf{Q} = \text{electromagnetic } \mathbf{Q},$$

in which n denotes a numerical quantity.

It is evident that there may be a particular set of fundamental units, for which n is 1; let them be $\mathbf{L}$, $\mathbf{M}$, $\mathbf{T}$. The multiplier for changing the electrostatic $\mathbf{Q}$ is $l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}$, and that for changing the electromagnetic $\mathbf{Q}$ is $l^{\frac{1}{2}}m^{\frac{1}{2}}$. Hence when we change from the units $\mathbf{L}$, $\mathbf{M}$, $\mathbf{T}$ to the units $\mathbf{L}'$, $\mathbf{M}'$, $\mathbf{T}'$ the equivalence changes to

$$l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1} \text{ electrostatic } \mathbf{Q}' = l^{\frac{1}{2}}m^{\frac{1}{2}} \text{ electromagnetic } \mathbf{Q}',$$

$$i.e., \quad lt^{-1} \text{ electrostatic } \mathbf{Q}' = \text{electromagnetic } \mathbf{Q}'.$$

Now lt^{-1} is the value of a velocity; and according to the theory of Clerk Maxwell,* it is the value of the velocity of light. It has been found as the result of a large number of experiments that the value ranges about that of the velocity of light in air, which is 3×10^{10} cm. per sec.

ART. 221.—**Coulomb; Ampere.** The C.G.S. systematic unit is defined by a special case of the above, namely

$$1\ \mathbf{Q}_{\text{c.g.s.}} \text{ per sec.} = \text{dyne per } \mathbf{P}_{\text{c.g.s.}} \text{ by (cm. radius)}^2 \text{ per cm. arc.}$$

At the International Congress of Electricians, which met at Paris in 1881, one-tenth of this electromagnetic unit of quantity was adopted as a convenient practical unit, and denominated a *coulomb*.

As it is not the unit of electricity but the unit of current which is directly measured, it is important to have a single word for denoting coulomb per second. At the Congress referred to, the term *ampere* was chosen for the purpose.

$$1 \text{ ampere} = \tfrac{1}{10}\ \mathbf{Q}_{\text{c.g.s.}} \text{ per sec.}$$

The practical units form a system in which $\mathbf{L}$ is 10^7 metres, $\mathbf{M}$ is 10^{-11} gramme, and $\mathbf{T}$ is the sound.

* *Electricity and Magnetism*, vol. II., chap. XX.

ART. 222.—**Unit of Electromotive Force; Volt.** The electromotive force of a circuit is the amount of work done on a unit of positive electricity in passing once round the circuit. It is expressed in terms of W per Q moved round; hence in the C.G.S. system, by erg per $\mathsf{Q}_{c.g.s.}$. The Congress adopted as a practical unit the unit which had been defined and adopted by the British Association, namely, 10^8 erg per $\mathsf{Q}_{c.g.s.}$, denominated the *volt.*

$$\text{Hence } 1 \text{ volt} = 10^8 \text{ erg per } \mathsf{Q}_{c.g.s.},$$
$$= 10^9 \text{ erg per coulomb.}$$

The volt, as will be seen from the following short table, is nearly equal to the electromotive force of a Daniell's cell.

A customary abbreviation for the term electromotive force is e.m.f.

ELECTROMOTIVE FORCE OF VOLTAIC CELLS.

Name of Cell.	Daniell.	Grove.	Bunsen.	Latimer-Clark.	Leclanché.
volts	1·12	1·95	1·85	1·46	1·42

ART. 223.—**Unit of Capacity; Farad.** We have seen that the idea of capacity is expressed in terms of Q per (W per Q). The ordinary C.G.S. unit is $\mathsf{Q}_{c.g.s.}$ per (erg per $\mathsf{Q}_{c.g.s.}$).

The Congress adopted the practical unit of the British Association, namely, the farad. It is defined by

$$1 \text{ farad} = \text{coulomb per volt,}$$
or
$$= 10^{-9}\, \mathsf{Q}_{c.g.s.} \text{ per (erg per } \mathsf{Q}_{c.g.s.}).$$

The microfarad, which is the one millionth part of the farad, is the most convenient unit for actual work.

A cable is an infinitely long cylindrical condenser. For a cable having a metallic core of a L radius, an insulating sheath of b L radius, and a specific inductive capacity k, the capacity per unit of length is

$$\frac{k}{2\,a \log \frac{b}{a}} \mathsf{Q} \text{ per } \mathsf{L} \text{ long} = \mathsf{W} \text{ per } \mathsf{Q}.$$

ART. 224.—**Unit of Resistance; Ohm.** When a steady current

of electricity flows round a circuit, the amount of the current is the same at every cross-section of the circuit. When the electromotive force is varied, the circuit being kept the same and at the same temperature, the amount of the current is found to be proportional to the electromotive force. Hence we have the law, discovered by Ohm,

$$k\ \mathbf{W} \text{ per } \mathbf{Q} = \mathbf{Q} \text{ per } \mathbf{T}.$$

This gives us the idea of *electric resistance.*

The C.G.S. unit is erg per $\mathbf{Q}_{c.g.s}$ per ($\mathbf{Q}_{c.g.s.}$ per sec.).

The practical unit, originated by the British Association and adopted by the Electrical Congress, is

volt per ampere.

The single equivalent term is *ohm;* so that

$$\begin{aligned} 1 \text{ ohm} &= \text{volt per ampere}, \\ &= 10^9 \text{ C.G.S. unit of resistance.} \end{aligned}$$

ART. 225.—**The Standard Ohm.** The British Association after defining the ohm appointed a committee to construct a standard which should realize the definition. The result of their measurements was that the ohm is represented by the resistance of a column of pure mercury at 0° C., one square millimetre in section and 105 centimetres long. In accordance with this result, standard coils were constructed of an alloy of two parts of silver to one of platinum, and issued to experimenters.

Subsequent measurements, made by various experimenters, agreed in showing that the standard ohm was slightly less than the ohm of the definition. The Paris Congress appointed a committee of electricians to make a fresh determination; and on their report the standard ohm has been authoritatively defined as the resistance of a column of mercury at 0° C., having a section of one square millimetre and a length of 106 centimetres.

The Siemens unit of resistance was defined as the resistance of a column of mercury at 0° C., having one sq. mm. in section, and 1 metre long.

ART. 226.—**Watt; Joule.** We have

1 $\mathbf{W}$ per $\mathbf{T}$ = ($\mathbf{W}$ per $\mathbf{Q}$ moved round) × $\mathbf{Q}$ passing round per $\mathbf{T}$.

A special unit of activity, called the *watt*, is obtained from this equivalence by making $\mathbf{W}$ per $\mathbf{Q}$ the volt, and $\mathbf{Q}$ per $\mathbf{T}$ the ampere. Thus

$$1 \text{ watt} = \text{volt by ampere,}$$
$$= 10^7 \text{ ergs per second.}$$

The *joule* is the corresponding unit of energy,

$$1 \text{ joule} = \text{volt by coulomb.}$$

These two denominations were proposed by Sir W. Siemens in his address to the British Association, 1882; they appear likely to be adopted, but they have not as yet the authoritative stamp of the other denominations defined in this section.

Also since

$$1\ \mathbf{W} \text{ per } \mathbf{T} = \{(\mathbf{W} \text{ per } \mathbf{Q}) \,/\, (\mathbf{Q} \text{ per } \mathbf{T})\} \times (\mathbf{Q} \text{ per } \mathbf{T}) \times (\mathbf{Q} \text{ per } \mathbf{T}),$$
$$= \mathbf{R} \times (\mathbf{Q} \text{ per } \mathbf{T})^2;$$
$$1 \text{ watt} = \text{ohm} \times (\text{ampere})^2.$$

The relation of the watt to the horse-power is

$$746 \text{ watts} = \text{horse-power.}$$

EXAMPLES.

Ex. 1. Find the multiplier for changing the electrostatic unit of potential from the centimetre, gramme, and second, to the metre, kilogramme, and second.

The old unit of potential is expressed by

$$\text{erg per } \mathbf{Q}_{\text{c.g.s.}}$$

Now $\quad 1 \text{ erg} = \text{gm. by cm. per sec. per sec. by cm.};$

and $\quad 1\ \mathbf{Q}^2_{\text{c.g.s.}} = \text{dyne by cm.}^2,$

$\quad = \text{gm. by cm. per sec. per sec. by cm.}^2$

But $\quad \cdot 01 \text{ metre} = \text{cm.},$

and $\quad \cdot 001 \text{ kgm.} = \text{gm.};$

hence, when we substitute instead of cm. and gm., we obtain

1 erg = ·001 × ·01 × ·01 kgm. by metre per sec. per sec. by metre,
= ·0000001 „ „ „

Also

$1\ \mathbf{Q}^2_{\text{c.g.s.}} = \cdot 001 \times \cdot 01 \times (\cdot 01)^2$ kgm. by metre per sec. per sec. by metre^2,
= ·000000001 „ „

$\therefore$

$$1\ \mathbf{Q}_{\text{c.g.s.}} = \frac{1}{10000\sqrt{10}}\sqrt{\text{kgm. by metre per sec. per sec. by metre.}}$$

Hence

$$\frac{\sqrt{10}}{10^3} \text{ kilogrammetre per } \mathbf{Q}_{\text{m.kg.s.}} = \text{erg per } \mathbf{Q}_{\text{c.g.s.}},$$

i.e., ·00316228 „ „ „

The kilogrammetre here meant is the absolute not the gravitational unit.

Ex. 2. Find the current in a circuit of 50 ohms, generated by a dynamo machine having an internal resistance of 5 ohms, when the electromotive force of the dynamo is 450 volts.

1 ohm = volt per ampere,

$\therefore$ 1 ampere = volt per ohm,

450 volts per (50 + 5)ohms ;

$$\therefore \quad \frac{450}{55} \text{ amperes,}$$

i.e., 8·2 amperes.

EXERCISE XLVIII.

1. Find the multipliers in the electrostatic system for changing the units of Quantity, Capacity, and Current from centimetre, gramme, and second, to metre, kilogramme, and second.

2. Find the corresponding multipliers when the units mentioned above belong to the electromagnetic system.

3. Find the multipliers for changing the electromagnetic units of Electromotive Force, Current, and Resistance, from the C.G.S. units to the F.P.S. units.

4. A table of electromotive forces is expressed in terms of the millimetre, milligramme, and second ; find the factor for changing to the C.G.S. unit.

5. Compare the millimetre-milligramme-second unit of current with the ampere.

6. Find the multiplier for changing cheval-vapeur to watt.

7. Compare the kilogrammetre with the joule.

8. If an electromotive force of 90 volts is maintained between the terminals of an incandescent lamp, and a current of 1·5 amperes flows through the lamp; what is the rate at which energy is supplied to the lamp?

9. A dozen incandescent lamps, each having a resistance of 2·75 ohms, are joined in a single circuit, and the resistance of the wires connecting the terminals with the terminals of the dynamo machine is 1·2 ohms. If the maximum electromotive force of the dynamo is 250 volts, what is the maximum current which can be sent through the lamps?

10. A single Grove's cell is employed to send a current through an external resistance of 100 ohms. What is the strength of the current taking the internal resistance of the cell at ·25 ohm?

11. Calculate, in terms of the watt, the activity of the above circuit.

12. When the poles of a battery were connected with the terminals of a tangent-galvanometer, a current of 24 amperes was produced; and when the resistance of the circuit was increased by 1 ohm, all else remaining as before, the strength of the current was 11 amperes. Find the electromotive force of the battery.

13. A circuit is formed containing galvanometer, battery, and connecting wires, the total resistance of the circuit being 4·85 ohms; the galvanometer shows a deflection of $48\frac{1}{2}^\circ$. When a piece of platinum wire is introduced into the circuit, the deflection falls to 29°. Calculate the resistance of the platinum wire, given $\tan 48\frac{1}{2}^\circ = 1{\cdot}121$, and $\tan 29^\circ = 0{\cdot}554$.

14. The ratio of the electrostatic to the electromagnetic unit of quantity is 3×10^{10} in the C.G.S. system; what is it in the F.P.S. system?

15. A battery of 50 Grove cells, having a total internal resistance of 13·5 ohms, is joined by a short-circuit; find the current which will be given.

16. The Board of Trade, acting under the Electric Lighting Act, have adopted a unit of energy which is defined as "the energy contained in a current of 1,000 amperes flowing under an electromotive force of one volt during one hour." Compare this unit with the joule.

SECTION XLIX.—RESISTANCE.

ART. 227.—**Resistance of a Substance.** When a steady current of electricity flows along a wire of uniform material, having a uniform cross-section, the strength of the current is directly

proportional to the difference of the electro-motive forces at the ends and to the cross-section, and inversely proportional to the length of the circuit, depending otherwise only on the nature and the temperature of the material of the wire. Hence we have a rate

k **Q** per **T** per **S** cross-section = (**W** per **Q**) difference per **L** length; it is called the *electric conductivity* of the substance. The idea is analogous to that of thermal conductivity; (**W** per **Q**) difference per **L** length takes the place of **Θ** difference per **L** normal; it may therefore be called the *gradient of electromotive force.*

The reciprocal is

$1/k$ (**W** per **Q**) diff. per **L** length = **Q** per **T** per **S** cross-section; it is called the *electric resistance* of the substance. It may be expressed in the equivalent form

$1/k$ (**W** per **Q**) diff. per (**Q** per **T**) curt. = **L** length per **S** cross-section.

The C.G.S. unit for the resistance of a substance is

erg per $\mathbf{Q}_{\text{c.g.s.}}$ per cm. per (erg per sec. per cm.^{2}).

The practical unit is

volt per cm. per (ampere per sq. cm.),

or volt per ampere = cm. per sq. cm.

It is equivalent to 10^9 of the above C.G.S. units.

ART. 228.—**Relative Resistance; Specific Resistance.** Let the electric resistance of a substance A be

r_1(**W** per **Q**) per **L** of A = (**Q** per **T**) per **S** of A;

this can be put into the form

r_1 (**S** per **L**) of A = (**Q** per **T**) per (**W** per **Q**).

Similarly for a substance B,

r_2 (**S** per **L**) of B = (**Q** per **T**) (per **W** per **Q**).

Hence r_2/r_1 (**S** per **L**) of B = (**S** per **L**) of A;

which denotes the resistance of B relatively to that of A.

If A is the standard substance (at a standard temperature) with which other substances are compared, then the relative resistance becomes the specific resistance. In the table below the standard

substance is mercury at 0° ; it is the most suitable substance for the purpose as the ohm is now defined in terms of a column of mercury (Art. 225).

Observe—The word "specific" is used throughout in the sense which it has in the term "specific gravity."

SPECIFIC ELECTRIC CONDUCTIVITY.

k **L** length per **S** section of substance at 0° = **L** length per **S** section of mercury at 0° C.

The resistance of mercury at 0° C. is ·943 ohm = metre per sq. mm. cross section.

SUBSTANCE.	k	SUBSTANCE.	k
Metal.		*Compound Metal.*	
Aluminium,	31	Brass, 30 Zn + 70 Cu,	13
Bismuth,	·8	German Silver,	3·8
Cadmium,	14	Steel,	6·7
Cobalt,	9·7	*Non-metal.*	
Copper,	54	Gas coke,	·015
Gold,	44	Glass at 200°	91×10^{-14}
Iron,	8·5	Phosphorus at 20°	69×10^{-8}
Lead,	4·9	*Liquids.*	$k \times 10^{-4}$
Lithium,	10·7	Nitric acid, 3 **V** acid per 7 **V** water,	·73
Magnesium,	23	Sulphuric acid, concentrated,	·12
Nickel,	7·4	Sulphate of copper, saturated solution,	·04
Palladium,	6·9	Sulphate of zinc, saturated solution,	·05
Platinum,	8·2	Water, 11 **V** water per **V** sulphuric acid,	·37
Silver,	62	Water, distilled,	·00000071
Tin,	8·9		
Zinc,	16·5		

ART. 229.—**Resistance in terms of Linear Density.** Let the resistance of a substance be

$$r \, \mathbf{S} \text{ per } \mathbf{L} = (\mathbf{Q} \text{ per } \mathbf{T}) \text{ per } (\mathbf{W} \text{ per } \mathbf{Q}).$$

Let its density be

$$\rho \, \mathbf{M} = \mathbf{S} \text{ by } \mathbf{L},$$

then ρ **M** per **L** = **S**.

Hence, by substituting for **S**,

$r\rho$ (**M** per **L**) per **L** = (**Q** per **T**) per (**W** per **Q**).

It is evident that **M** per **L** denotes the unit of linear density. The relative resistance becomes

$r_2\rho_2/r_1\rho_1$ (**M** per **L**) per **L** of B = (**M** per **L**) per **L** of A.

Art. 230.—**Multiple Circuit.** Let the points P and Q of a circuit (Fig. 27) be connected by three wires A, B, and C, the respective resistances of which are r_1 ohm, r_2 ohm, r_3 ohm. Then

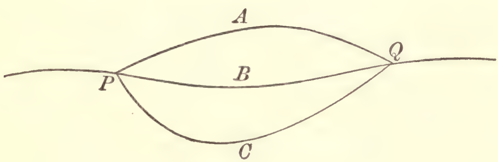


Fig. 27

the conductivity of A is $1/r_1$ mho, of B $1/r_2$ mho, and of C $1/r_3$ mho. The collective conductivity is the sum of the conductivities of the different wires; hence it is

$$\frac{1}{r} + \frac{1}{r_2} + \frac{1}{r_3} \text{ mho,}$$

$$i.e., \quad \frac{r_2r_3 + r_3r_1 + r_1r_2}{r_1r_2r_3} \text{ mho.}$$

The word-symbol *mho* is used by Sir W. Thomson to denote the reciprocal unit to the ohm, that is, for ampere per volt.

The resulting resistance is the reciprocal of the collective conductivity; hence it is

$$1/\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) \text{ ohm,}$$

$$i.e., \quad \frac{r_1r_2r_3}{r_2r_3 + r_3r_1 + r_1r_2} \text{ ohm.}$$

If the three resistances are equal, the collective conductivity is $3/r$ mho, and the resulting resistance $r/3$ ohm. This is the case

when three similar cells are joined in multiple arc; and, generally, when n similar cells are so joined, the resulting resistance of the battery part of the circuit is one nth of the resistance of a single cell.

Compare the ideas of Arts. 104 and 105.

ART. 231.—**Wheatstone's Bridge.** This name is applied to a special arrangement of a battery circuit by means of which the resistance of a wire may be determined (Fig. 28). The terminals

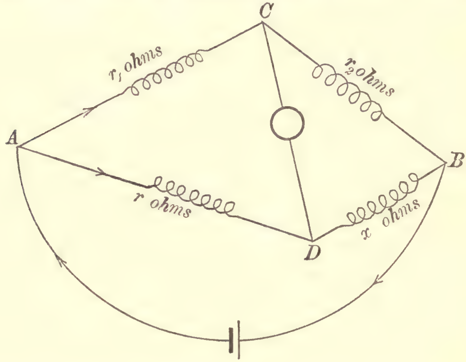


Fig. 28

A, B of the battery are connected by a two-fold circuit, each branch of which contains two resistances; and the points of junction, C, D, are connected by the galvanometer wire. The resistance of each of the wires AC and CB is known, the resistance of AD can be varied by altering its length, and DB is the wire of unknown resistance. The length of AD is varied until no current passes through the galvanometer.

For any length of the circuit the value k of the rate

$$k \text{ volts fall} = \text{ohm of length}$$

is constant. That is one way of stating Ohm's Law. Let e volts be the fall of potential in passing from A to B, then for the one branch

$$\frac{e}{r_1 + r_2} \text{ volts fall} = \text{ohm};$$

and for the other branch

$$\frac{e}{r+x} \text{ volts fall} = \text{ohm}.$$

Hence the fall from A to C is $r_1e/(r_1+r_2)$ volts, and that from A to D is $re/(r+x)$ volts. When there is no current through the galvanometer, these falls of potential must be equal; therefore we get the equation

$$\frac{r_1e}{r_1+r_2} = \frac{re}{r+x},$$

from which

$$x = \frac{r_2}{r_1}r.$$

Multipliers for Changing from one System of Electrical Units to Another.

Fundamental Equivalences—l new **L** = old **L**; m new **M** = old **M**; t new **T** = old **T**.

Derived Equivalence—n new unit = old unit.

Unit.	Electrostatic Kind.	Electromagnetic Kind.
	n	n
Magnetic pole, ...		$l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-1}$
Intensity of magnetic field, .		$l^{-\frac{1}{2}} m^{\frac{1}{2}} t^{-1}$
Magnetic potential, . . .		$l^{\frac{1}{2}} m^{\frac{1}{2}} t^{-1}$
Magnetic moment, . . .		$l^{\frac{5}{2}} m^{\frac{1}{2}} t^{-1}$
Intensity of magnetization, .		$l^{-\frac{1}{2}} m^{\frac{1}{2}} t^{-1}$
Quantity of electricity, . .	$l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-1}$	$l^{\frac{1}{2}} m^{\frac{1}{2}}$
Intensity of electric field, . .	$l^{-\frac{1}{2}} m^{\frac{1}{2}} t^{-1}$	$l^{\frac{1}{2}} m^{\frac{1}{2}} t^{-2}$
Electric potential, . . .	$l^{\frac{1}{2}} m^{\frac{1}{2}} t^{-1}$	$l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-2}$
Electric density, . . .	$l^{-\frac{1}{2}} m^{\frac{1}{2}} t^{-1}$	$l^{-\frac{3}{2}} m^{\frac{1}{2}}$
Electric capacity, . . .	l	$l^{-1} t^{2}$
Specific inductive capacity, .	1	$l^{-2} t^{2}$
Current, ...	$l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-2}$	$l^{\frac{1}{2}} m^{\frac{1}{2}} t^{-1}$
Resistance, ...	$l^{-1} t$	$l t^{-1}$
Conductivity of a substance referred to length and section,		$l^{-2} t$
Conductivity of a substance referred to length and line-density,		$l m^{-1} t$

EXAMPLES.

Ex. 1. What is the relative resistance of two copper wires, the one 10 feet long and weighing 35 grammes, the other 6 feet long and weighing 10·5 grammes?

Let the resistance of the copper be

$$c \text{ ohm} = (\text{ft. long})^2 \text{ per gramme.}$$

For the former piece

$$100/35 \text{ ft}^2. \text{ per gm.};$$

$$\therefore \quad 100c./35 \text{ ohms.}$$

For the latter piece

$$36/10{\cdot}5 \text{ ft}^2. \text{ per gm.};$$

$$\therefore \quad 36c./10{\cdot}5 \text{ ohms.}$$

Hence $\dfrac{100 \times 10{\cdot}5}{35 \times 36}$ ohms in former = ohm in latter,

i.e., 5/6 ohms in former = ohm in latter.

Ex. 2. Two points, A and B, are joined by three wires in multiple arc, the resistances of which are 3 ohms, 5 ohms, 7 ohms. What is the resulting resistance between A and B?

$$1 \text{ ohm} = \text{volt per ampere.}$$

For the 1st wire the resistance is 3 ohms,

$\therefore$ the conductivity is $\frac{1}{3}$ ampere per volt.

The conductivity for the 2nd is $\frac{1}{5}$ ampere per volt,
and for the 3rd, $\frac{1}{7}$ ampere per volt;
hence the total conductivity is $\frac{1}{3} + \frac{1}{5} + \frac{1}{7}$ ampere per volt,

i.e., $\dfrac{5 \times 7 + 7 \times 3 + 3 \times 5}{3 \times 5 \times 7}$ ampere per volt,

i.e., $\dfrac{71}{105}$ ampere per volt.

Hence the resulting resistance is

$$\frac{105}{71} \text{ volt per ampere,}$$

i.e., 1·5 ohm.

Ex. 3. Find the arrangement of 100 equal cells which will give

the greatest current in a conductor whose resistance is 25 times that of one cell.

Take for unit of electromotive force that of one cell and denote it by **E**, and for unit of resistance that of one cell and denote it by **R**. Let **C** denote a unit of current such that

$$1\ \mathbf{C} = \mathbf{E} \text{ per } \mathbf{R}.$$

First, for the arrangement *in series.* The electromotive force is 100 **E**, and the resistance is $100 + 25$ **R**; therefore the current is 100/125 **C**.

Second, for the arrangement *in multiple arc.* The electromotive force is 1 **E**, and the resistance $\cdot 01 + 25$ **R**; hence the current is 100/2501 **C**.

The ratio of the current given by the former arrangement to that given by the latter is 2501 to 125.

Ex. 4. Determine the electric resistance of the material of a wire 437 millimetres long, which has a resistance of ·1257 ohms, and which weighs 411 milligrammes in air, and 365 milligrammes in water.

The resistance of a material is measured in terms of

ohm = cm. long per sq. cm. cross-section.

For the material in question

$$\cdot 1257 \text{ ohm} = 43{\cdot}7 \text{ cm. long per } x \text{ sq. cm.},$$

i.e., $$\frac{\cdot 1257 \times x}{43{\cdot}7} \text{ ohm} = \text{cm. per sq. cm.}$$

To find x.

$$\frac{411}{411 - 365} \text{ gm. per cm. long} = \text{sq. cm. section},$$

$$\cdot 411 \text{ gm. per } 43{\cdot}7 \text{ cm. long},$$

$$\therefore \quad \frac{\cdot 411 \times 46}{43{\cdot}7 \times 411} \text{ sq. cm.}$$

Hence $$\frac{\cdot 1257 \times \cdot 411 \times 46}{43{\cdot}7 \times 43{\cdot}7 \times 411} \text{ ohm per (cm. per sq. cm.)},$$

i.e., 3·028 microhms per (cm. per sq. cm.).

Ex. 5. The resistance of the wire of a galvanometer is 4,000 ohms; it is required to find the resistance of a wire, which, acting as a shunt, will reduce ten-fold the sensitiveness of the galvanometer.

Let it be x ohms. The conductivity of the galvanometer is $1/4000$ ampere per volt, of the shunt $1/x$ ampere per volt; therefore of the two together $1/4000 + 1/x$ ampere per volt. Hence the ratio of the current through the galvanometer to the total current will be

$$\frac{1}{4000} \text{ ampere through galvanometer} = \frac{1}{4000} + \frac{1}{x} \text{ ampere total,}$$

i.e., x ampere through galvanometer $= 4000 + x$ ampere total.
But this is given to be 1 to 10; hence

$$10\,x = 4000 + x,$$

$$x = 444.$$

Ans.—444 ohms.

EXERCISE XLIX.

1. Compare the currents which the same electromotive force is capable of producing in two wires of the same material, whose lengths are as 5 to 1, and cross-sections as 3 to 2.

2. The resistance of a piece of platinum wire, 41 metres long, and ·5 millimetres in diameter, is found to be ·19 ohms. What is the resistance of a bar of the same material 1 decimetre long and 1 square centimetre in section?

3. Compare the resistances of two copper wires, one of them 8 feet long and weighing 1/4 ounce, the other 14 feet long and weighing 6/7 ounce.

4. A piece of copper wire 100 yards long weighs 1 lb.; another piece of copper wire 500 yards long weighs 1/4 lb. Find the relative resistance of the latter piece to the former.

5. Find the resistance of 485 m. of copper wire, one mm. in diameter, at 0° C.

6. Find the resistance per mile of iron wire, ·24 inch in diameter.

7. What length of German silver wire one mm. in diameter will give a resistance of one ohm?

8. What is the resistance of 2000 yards of German silver wire, ·0108 inch in diameter?

9. Find the length of a copper wire 1 mm. in diameter which has the resistance of one C.G.S. electromagnetic unit.

10. The resistance of a column of silver 100 cm. long and 1 gm. in weight is ·1687 ohms. Find the resistance of silver in terms of the C.G.S. unit.

11. Two brass plates, 10 inches square, are separated from one another by a plate of gutta-percha 1/4 inch in thickness. How many miles of copper wire, ·056 inch in diameter, will have a resistance equal to that between the plates. The relative resistance of guttapercha to copper is $2 \cdot 8 \times 10^{20}$?

12. An electrical current may pass from A to B by either of the wires ACB and ADB, the resistances of which are 3 ohms and 7 ohms respectively. What is the resistance of a single wire, which can replace ACB and ADB in such a way as not to produce any alteration in the current in the rest of the circuit?

13. With the shunts 1/999, 1/99, 1/9, compare the currents in the galvanometer, when any two of the shunts, and when all three are in circuit together.

14. In a submarine cable 1000 knots in length, the electrical resistance of the conductor is 5 ohms per knot, and the whole insulation resistance of the gutta-percha sheath is 115,000 ohms. Determine the total resistance of the cable exclusive of batteries.

15. Three incandescent lamps having a resistance of 50 ohms each, are joined in multiple arc. What is the resulting resistance?

16. Two wires, whose conductivities, lengths, and cross-sections are as 7 to 6, 5 to 3, 2 to 1 respectively, are in the same circuit. Compare the rate at which heat is developed in the former to the rate at which heat is developed in the latter.

17. A galvanometer of 500 ohms is shunted by a shunt of 50 ohms. Compare the amounts of heat generated in the galvanometer and shunt.

18. Show how to arrange 12 similar galvanic cells, each of which has a resistance of 1·2 units, so as to give the current of greatest strength through a wire whose resistance is 2·5 units.

19. A battery of 12 similar cells is connected in series; each cell has an electromotive force of 1·1 volt, and a resistance of 3 ohms; and the resistance of the external circuit is 240 ohms. What is the strength of the current?

20. If there are 20 cells in a battery, each having a resistance of 2 ohms, and if the external resistance is 1 ohm, what arrangement of cells will give the strongest current?

21. Find the condition which must hold, when the current given by the arrangement in series is equal to the current given by the arrangement in multiple arc.

22. Find the current when a battery of 12 cells, each having a resistance of 100 ohms and an electromotive force of 1·5 volts, is joined to an external circuit of 1000 ohms; *first*, when the cells are arranged in multiple arc; *second*, in series; *third*, in two series joined in multiple arc.

23. A current from a battery of 3 Grove's cells, having each an electromotive force of 1·8 volts, and resistance of ·8 ohms, is passed through a piece of platinum wire 1·5 metres long and 0·12 mm. in diameter for one minute. Find the amount of heat developed in the wire.

24. The terminals of a battery formed of seven Daniell's cells in series are joined by a wire 35 feet long. One binding screw of a galvanometer is joined by a wire to the copper of the third cell, reckoning from the copper end. With what point of the 35-feet wire can the other screw of the galvanometer be connected so that the needle shall not be deflected?

25. A table of electric conductivities is expressed in terms of the millimetre, milligramme, and second. Find the multiplier for changing to the C.G.S. unit.

CHAPTER SEVENTH.

ACOUSTICAL.

SECTION I.—MUSICAL SOUND.

Art. 233.—**Period and Frequency.** It is a property of a vibrating body that it vibrates always in the same amount of time, whether the amplitude of its vibration is large or small, provided that the amplitude does not exceed certain limits, which differ for different bodies. For instance, if an iron bar be held in a vice, and the upper end be displaced from the perpendicular, the bar, when let go, will vibrate on either side of the perpendicular, each point in the bar performing a simple harmonic motion (Art. 123). Each succeeding vibration has a less amplitude than its predecessor, but the time occupied in making the vibration remains the same. This constant time is called the *period* of the body, and is expressed in terms of

$$\mathsf{T} \text{ per vibration.}$$

The vibration may be defined in one or other of two ways; either as the movement from one side to the other, or the movement from one side to the other and back again. The latter is the more appropriate definition; for distinction it is sometimes called a complete vibration.

The reciprocal idea is the *frequency*; it is expressed in terms of

$$\text{vibrations per } \mathsf{T}.$$

Art. 234.—**Wave-length.** A disturbance initiated at any part

of a material medium is propagated outwards in all directions and with a constant velocity, provided the medium is uniform.

Let the velocity of propagation be

v **L** per **T**,

and the frequency n vibrations per **T**;

then, by eliminating the time-unit, we get

v/n **L** per vibration.

This gives us the idea of *wave-length*. The wave-length is the distance-period, that is the uniform distance from one point of greatest condensation to the next point of greatest condensation.

The reciprocal is

n/v vibrations per **L**.

ART. 235.—**Pitch.** The pitch of a sound depends on the number of vibrations received by the ear per unit of time. It is the same as the frequency, when the spectator and the source of sound are at rest relatively to one another.

If the spectator and the vibrating body move towards one another with a velocity v_1 **L** per **T**, the velocity with which the vibrations will arrive will be $v + v_1$ **L** per **T**; and as there are n/v vibrations per **L**, he will receive $n(v + v_1)/v$ vibrations per **T**.

If they move from one another with a velocity v_1 **L** per **T**, the velocity with which the vibrations will arrive will be $v - v_1$ **L** per **T**, and the spectator will receive $n(v - v_1)/v$ vibrations per **T**.

ART. 236.—**Intensity.** By the objective intensity of a source of sound is meant the amount of energy transformed per unit of time. It is expressed in the form

μ **W** per **T**.

Its amount at any time is proportional to the square of the amplitude of the vibrations.

By the intensity of the sound at a given position in the medium is meant the amount of energy received per unit of time per unit of cross-section; it is expressed in terms of

W per **T** per **S** cross-section.

Suppose concentric spheres drawn round the source of sound. If there be no absorption, the same amount will pass across each, namely,

$$\mu \, \mathsf{W} \text{ per } \mathsf{T}.$$

Now, by Art. 87,

$$4\pi \, \mathsf{S} \text{ spherical surface} = (\mathsf{L} \text{ radius})^2,$$

hence the intensity is given by

$$\mu \, \mathsf{W} \text{ per } \mathsf{T} = 4\pi \, \mathsf{S} \text{ per } (\mathsf{L} \text{ radius})^2,$$

$$\textit{i.e.,} \qquad \frac{\mu}{4\pi} \mathsf{W} \text{ per } \mathsf{T} = \mathsf{S} \text{ cross-section per } (\mathsf{L} \text{ radius})^2,$$

$$\text{or} \qquad \frac{\mu}{4\pi} \mathsf{W} \text{ per } \mathsf{T} = \text{steradian}.$$

The last form shows that the current through a constant cross-section is proportional to the solid angle subtended by the cross-section.

The intensity perceived by the ear is the objective intensity per unit of cross-section, modified by difference in sensitiveness to sounds of different wave-lengths.

EXAMPLES.

Ex. 1. Find the wave-length in air of a note making 50 vibrations per second, taking the velocity of propagation in air at 1100 feet per second.

$$50 \text{ vibrations} = \text{sec.},$$

$$1100 \text{ feet} = \text{sec.};$$

$$\therefore \qquad \frac{1100}{50} \text{ feet} = \text{vibration},$$

$$\textit{i.e.,} \qquad 22 \text{ feet} = \text{vibration}.$$

Ex. 2. An express train rushes past a station at the speed of 40 miles an hour, and blowing a whistle, the frequency of which is 1000 vibrations per second. What will be the difference in pitch of the notes heard by a spectator as the train comes up and

goes away? The velocity of sound in air is then 1090 feet per second.

Velocity of train $\frac{40 \times 22}{15}$ feet per second,

i.e., 60 feet per second approximately.

Hence on coming up 60 feet additional per second,

and on going away 60 feet less per second;

but $\frac{1000}{1090}$ vibrations per foot,

$\therefore$ $\frac{1000 \times 120}{1090}$ vibrations difference per second,

i.e., 110 vibrations difference per second.

EXERCISE L.

1. Find the wave-length of a note making 1,000 vibrations per second, both in air and in water; the velocity of sound in air being 1,100 ft. per sec., and in water 4,900 ft. per sec.

2. A tuning-fork makes 256 vibrations per second, and the velocity of sound is 340 metres per second; what is the value of the wave-length of the note produced?

3. It is observed that $6\frac{1}{2}$ seconds elapse between the flash and report of a lightning discharge. At what distance did the discharge take place?

4. A stone is dropped down a well, and is heard to strike the bottom after an interval of 3 seconds; determine the depth, the velocity of sound being 1,140 ft. per sec.

5. Three observers are stationed, the first at a mile, the second at two miles, and the third at three miles from a gun. At 12 o'clock precisely the gun is fired; state the times at which the explosion will be heard at the several stations.

6. Taking 1,120 ft. per sec. as the velocity of sound in air, find the number of vibrations which a middle C tuning-fork (which vibrates 264 times per second) must make before its sound is audible at a distance of 154 feet.

7. A shot is fired at 500 yards, and a man standing at a distance of 100 yards from the target hears the reports of the firing and of the impact at the same time. The time of flight of the shot is 1 sec.; find the distance of the man from the place of firing.

8. A locomotive moving at 100 ft. per sec. carries a steam-whistle which produces 1,000 vibrations per second. What will be the pitch of the note heard by a person standing close to the rails before and after the locomotive has passed? Velocity of sound as in question 1.

SECTION LI.—VELOCITY OF SOUND.

ART. 237.—**Extensibility of a Solid.** Suppose that a bar of a substance of uniform cross-section is subjected to an equal pull at either end. The effect will be an alteration of the length of the bar; and so long as the stretching force is not sufficiently great to strain the bar beyond the limit of elasticity, the extension is proportional to the force. The bar, in its strained state, exerts a force equal and opposite to the stretching force. The relation between the effect and the cause is expressed in the form

$$\epsilon \text{ L increment per L original} = \text{F per S}.$$

This is called the *extensibility* of the solid. The reciprocal idea is

$$1/\epsilon \text{ F per S} = \text{L increment per L original};$$

it is called Young's modulus, or the modulus of elasticity, or the resilience due to longitudinal extension.*

The rate of longitudinal compression, when the bar is subjected to an equal compressing force at either end, is expressed in terms of

$$\text{L decrement per L original} = \text{F per S}.$$

Its value is the same as that of the extensibility.

ART. 238.—**Compressibility of a Solid or Liquid.** When a solid or liquid is subjected to an equal increment of pressure all over the surface, the change of volume produced per unit of original volume is proportional to the additional pressure. The relation between the effect and the cause is expressed in the form

$$c \text{ V decrement per V original} = (\text{F per S}) \text{ increment}.$$

This is called the *compressibility* of the substance. The reciprocal is

$$1/c \ (\text{F per S}) \text{ increment} = \text{V decrement per V original};$$

it is called the modulus of compressibility, or (by Everett) the resilience due to hydrostatic pressure.

The expansibility of a solid or liquid is expressed in terms of

$$\text{V increment per V original} = \text{F per S}.$$

Its value is the same as that of the compressibility.

* Everett, *Units and Physical Constants*, p. 47.

ART. 239.—**Compressibility of a Gas.** In the case of a gas one value of the compressibility is true only for a small range of increase of pressure. Let the original pressure of a portion of gas be p **F** per **S** and its volume v **V**, and let the pressure be changed by a small amount δp **F** per **S**. The volume will in consequence, there being no change of temperature, suffer a decrement δv **V**. Now by Boyle's Law

$$pv \text{ F per S by V} = (p + \delta p)(v - \delta v) \text{ F per S by V},$$

$$\therefore \quad pv = pv + \delta p \times v - p \times \delta v - \delta p \times \delta v.$$

As δv and δp are each small fractions, their product $\delta p \times \delta v$ will be still smaller, and may be left out of account. Hence

$$\frac{\delta v}{v \times \delta p} = \frac{1}{p}.$$

Now $\frac{\delta v}{v \times \delta p}$ is the value of **V** decrement per **V** original = **F** per **S**;

$\therefore \frac{1}{p}$ is the value of the compressibility, and p the value of the modulus of elasticity.

Observe that δv and δp each denote one numerical value; the δ is used to denote that the value is small.

ART. 240.—**Velocity in a Solid or Liquid.** In a solid medium the square of the velocity of sound is directly proportional to the modulus of elasticity and inversely proportional to the density. Thus

k (**L** per **T**)^2 = **F** per **S** per (**L** inct. per **L** origl.) per (**M** per **V**).

If **F**, **S**, and **V** are each systematic units, then

1 (**L** per **T**)^2 = (**M** by **L** per **T** per **T**) per **L**^{2} per(**L** inct. per **L** origl.) per (**M** per **L**^{3}).

The dimensions of the right-hand expression are the same as those of the left-hand expression.

For a liquid we have the same law, only compressibility is substituted for extensibility; thus

k (**L** per **T**)^2 = **F** per **S** per (**V** inct. per **V** origl.) per (**M** per **V**);

and k is equal to 1, when all the units are systematic.

ART. 241.—**Velocity in a Gas.** The same law modified holds for a gas. The modulus of elasticity of a gas has the same value as the pressure (Art. 239); but the law

$$1\ (\mathsf{L}\text{ per }\mathsf{T})^2 = (\mathsf{F}\text{ per }\mathsf{S})\text{ total per }(\mathsf{M}\text{ per }\mathsf{V})$$

does not hold, though the units are systematic, on account of the development of heat during the compression, by means of which the sound is propagated. The true law is

$$\gamma\ (\mathsf{L}\text{ per }\mathsf{T})^2 = (\mathsf{F}\text{ per }\mathsf{S})\text{ total per }(\mathsf{M}\text{ per }\mathsf{V}),$$

where γ denotes the value of the ratio of the specific heat of the gas at constant pressure to the specific heat at constant volume (Art. 184); namely 1·408. The unmodified law is called Newton's Rule.

Since for a gas we have

$$R\ \Theta\text{ total} = (\mathsf{F}\text{ per }\mathsf{S})\text{ total per }(\mathsf{M}\text{ per }\mathsf{V}),$$

the total temperature being measured from the absolute zero, by means of the above law we deduce

$$\gamma\ (\mathsf{L}\text{ per }\mathsf{T})^2 = R\ \Theta\text{ total},$$

or $$\gamma/R\ (\mathsf{L}\text{ per }\mathsf{T})^2 = \Theta\text{ total}.$$

The velocity of sound in dry air at 0° C. is 332 metres per second.

ART. 242.—**Frequency of a Stretched String and of an Organ Pipe.** The square of the velocity with which a disturbance travels along a stretched string is directly proportional to the stretching force, and inversely proportional to the line-density. If the units are systematic,

$$1\ (\mathsf{L}\text{ per }\mathsf{T})^2 = \mathsf{F}\text{ per }(\mathsf{M}\text{ per }\mathsf{L}),$$

that is, $$= (\mathsf{M}\text{ by }\mathsf{L}\text{ per }\mathsf{T}\text{ per }\mathsf{T})\text{ per }(\mathsf{M}\text{ per }\mathsf{L}).$$

The disturbance has to travel twice the length of the string before it makes a complete vibration; hence

$$2\ \mathsf{L}\text{ per vibration} = \mathsf{L}\text{ string}.$$

This is the fundamental wave-length. The string can also make a vibration in one half, one third, one fourth, etc., of the fundamental wave-length. The frequencies corresponding to these wave-lengths are called the *harmonics* of the fundamental frequency.

In the case of an open organ pipe,

$$2\ \mathsf{L} \text{ per vibration} = \mathsf{L} \text{ length of pipe},$$

and when the pipe is stopped,

$$4\ \mathsf{L} \text{ per vibration} = \mathsf{L} \text{ length of pipe}.$$

Examples.

Ex. 1. Water is diminished 1/21000 of its bulk by an additional atmosphere of pressure. Find the modulus of compressibility.

$$1/21000\ \mathsf{V} \text{ decrement per } \mathsf{V} = \text{atmosphere additional},$$

$$\therefore \quad 21000 \text{ atmospheres} = \mathsf{V} \text{ per } \mathsf{V},$$

$$76 \times 13{\cdot}6 \text{ gm. weight per sq. cm.} = \text{atmosphere};$$

$$\therefore \quad 21000 \times 76 \times 13{\cdot}6 \text{ gm. weight per sq. cm.} = \mathsf{V} \text{ per } \mathsf{V},$$

i.e., $2{\cdot}16 \times 10^7$ gm. weight per sq. cm. $= \mathsf{V}$ per V.

or $21{\cdot}6$ gm. weight per sq. cm. $= 10^{-6}\ \mathsf{V}$ decrement per V.

Ex. 2. The velocity of sound being 33300 cm. per sec. at 0° C. and 760 mm. pressure; what is it when the temperature is 25° C. and the pressure 745 mm.?

It is independent of the change of pressure.

$$33300^2 \text{ (cm. per sec.)}^2 = 273 \text{ deg. Cent.},$$

$$273 + 25 \text{ deg. Cent.},$$

$$\therefore \quad \frac{273+25}{273}\, 33300^2 \text{ (cm. per sec.)}^2;$$

$$\therefore \quad 33300\sqrt{\frac{298}{273}} \text{ cm. per sec.};$$

i.e., 34800 cm. per sec.

Ex. 3. The mass of a cubic foot of wrought iron is 480 lbs. The velocity of sound in an iron bar is 17,000 feet per sec. Find the force required to elongate an iron bar of one square inch sectional area by 1/10000 of its original length.

Let the modulus be

μ lb. by ft. per sec. per sec. per sq. ft. per (ft. inct. per ft. origl.);

the density is 480 lb. per cubic foot; therefore (Art. 240) the velocity of sound along the bar is

$$\sqrt{\frac{\mu}{480}} \text{ feet per sec.}$$

But this is 17,000 feet per sec.,

$$\therefore \qquad \mu = 17000^2 \times 480.$$

Hence $\quad 17000^2 \times 480$ poundals = sq. ft. by (ft. per ft.),

$$\tfrac{1}{144} \text{ sq. ft. by } (\tfrac{1}{10000} \text{ ft. per ft.});$$

$$\therefore \qquad \frac{17000^2 \times 480}{144 \times 10000} \text{ poundals.}$$

Or at a place where gravity is 32·2 ft. per sec. per sec.,

$$\frac{17000^2 \times 480}{144 \times 10000 \times 32{\cdot}2} \text{ pound-weight,}$$

i.e., $\quad$ 2992 pound-weight.

Ex. 4. Calculate the lowest number of vibrations per second of a pianoforte wire 6 feet long, $\frac{1}{4}$ oz. to the foot, stretched by a tension of 10 lbs. weight.

The line-density is $\dfrac{1}{4 \times 16}$ lb. per foot,

The stretching force is

$$10 \times 32{\cdot}2 \text{ lb. by ft. per sec. per sec.;}$$

$\therefore$ the velocity of the disturbance along the wire is

$$\sqrt{10 \times 32{\cdot}2 \times 4 \times 16} \text{ ft. per sec.,}$$

i.e., $\quad 8\sqrt{322}$ ft. per sec.

Now $\quad$ 2 ft. per vibration = ft. wire,

6 ft. wire;

$\therefore \quad$ 12 ft. per vibration;

$$\therefore \qquad \frac{8\sqrt{322}}{12} \text{ vibrations per sec.;}$$

$$\textit{i.e.,} \qquad \frac{2 \times 17{\cdot}94}{3} \text{ vibrations per sec.,}$$

i.e., $\quad$ 12 vibrations per sec.

EXERCISE LI.

1. The velocity of sound in air at 0° C. is 333 metres per second ; what is it at 27° C.?

2. One o'clock is signalled by a time-gun, at what time will the signal be heard by a person at the distance of one mile, the temperature being 70° F.?

3. Assuming the velocity of sound in air to be 1,120 ft. per sec., determine the length of the wave produced in air by a tuning-fork which vibrates 384 times per second. Determine also the length of an open organ pipe which would yield the same note as the tuning-fork.

4. The length of the column of air which resounds most freely to a given tuning-fork is 32·2 cm.; if the velocity of sound in air be 33,000 cm. per second, determine the frequency of the fork.

5. Find the number of vibrations made in a second by an organ pipe 16 ft. long, open at both ends, sounding its fundamental note, the velocity of sound in air being 1,100 ft. per sec.

6. When the velocity of sound in air is 1,100 ft. per sec., how many vibrations per second are made by a closed organ pipe 4 feet long, sounding its fundamental note ?

7. Calculate the frequency of an open organ pipe 3 ft. long sounding its fundamental note, when the temperature is such that the velocity of sound is 1,092 ft. per sec. By how many would the frequency be increased by a rise of temperature of the air to the extent of 2·74 deg. Cent.

8. A pianoforte wire, 5 ft. long, which weighs 14 lbs. per nautical mile, is stretched with a pull of 100 lbs. Find the number of vibrations made per second when it is sounding its lowest note.

9. Supposing, at any particular time and place, the pressure of the atmosphere to be $14\frac{1}{2}$ lbs. to the square inch, and a cubic foot of it to weigh 536 grains, and the intensity of gravity to be 32·2 ft. per sec. per sec., what would be the velocity of sound in air, there and then, according to Newton's rule.

CHAPTER EIGHTH.

OPTICAL.

SECTION LII.—PHYSICAL.

ART. 243.—**Light.** The term *light* may be used in two senses; first, as equivalent to radiant energy; second, to denote that species of radiant energy which affects our organs of vision. Both of these quantities are properly measured in terms of **W**, the unit of energy, or rather, in terms of **W** per **T**, the unit for current of energy. In measuring the subjective quantity it is difficult to give proper weight to the preference exercised by the organs of vision for particular kinds of radiation.

ART. 244.—**Intensity of a Source.** Suppose that we have a single source of light placed in a homogeneous medium, and that it is sending forth a constant current

$$\mu \, \mathbf{W} \text{ per } \mathbf{T}.$$

The light will be propagated in straight lines and equally in all directions. Suppose a spherical surface drawn round the source, having the source for centre. Any small portion of this surface will be cross-sectional to the direction of the current at the place, and the whole amount of cross-section is given by (Art. 87)

$$4\pi \, \mathbf{S} \text{ cross-section per } (\mathbf{L} \text{ distance})^2.$$

The total current crossing this surface is the same as that emitted by the source, provided none of the light is absorbed by the

medium. Hence the current per unit of cross-section at unit distance, or the intensity of the source at unit distance is given by

$$\mu/4\pi \, \mathsf{W} \text{ per } \mathsf{T} = \mathsf{S} \text{ cross-section per } (\mathsf{L} \text{ distance})^2,$$
$$= \text{steradian (Art. 88).}$$

ART. 245.—**Illuminating Power.** By the illuminating power of a source of light is meant the relative intensity as judged by the human organs of vision. If two sources emit the same kind of light, that is, light of the same colour, their illuminating powers are strictly comparable. The comparison is commonly effected by finding the distances at which they produce equal illuminating effects as judged by the eye.

When two sources emit white light, their illuminating powers are also comparable, and the more so the greater the agreement in composition of the two lights.

ART. 246.—**Standards of Light.** Hitherto it has been the custom to measure illuminating power by means of a standard source. The English standard source is a sperm candle burning 120 grains per hour, six of the candles weighing one pound. The French standard source is a Carcel burner; it is equal to 9·3 standard candles.

The Paris Congress of Electricians appointed a committee to report on a standard of light, and recently the following standard was adopted. The unit of each kind of simple light is the quantity of light of the same kind emitted in a normal direction by a square centimetre of surface of molten platinum at the temperature of solidification. The practical unit of white light is the quantity of light emitted normally by the same source.

ART. 247.—**Luminosity; Brightness.** The luminosity of an incandescent surface may be defined as the amount of current of light emitted per unit of surface.

The brightness of an object is estimated by the light received

on the surface of the eye when directed towards the object per unit of solid angle (Art. 88), subtended by the object at the eye.*

The term brightness is also used to denote the subjective intensity of the current of light received from a small luminous body, such as a star or planet.

ART. 248.—**Wave-length.** The kinds of radiant energy are distinguished by different values of wave-length. The class of visible rays have smaller wave-lengths than the class of heat rays, and the photographic rays have smaller wave-lengths than the visible rays. The velocity of propagation in space is the same for all, namely, 300 million metres per second according to the most recent experiments. From the wave-length and the velocity the frequency can be deduced as in the case of sound (Art. 234).

WAVE-LENGTHS IN AIR FOR THE LINES IN THE SOLAR SPECTRUM.

λ millionths of a millimetre per wave.

Name of Line.	Substance.	λ	Name of Line.	λ
Visible portion of Spectrum.			*Ultra-Violet portion.*	
A	Atmospheric	760	L	382
B	Atmospheric	687	M	373
C	Hydrogen	656	N	358
D_1	Sodium	590	O	344
D_2	Sodium	589	P	336
E	Iron	527	Q	329
b	Magnesium	517	R	318
F	Hydrogen	486	S	310
G	Iron	431	T	302
h	Hydrogen	410	U	295
H	Calcium	397	*Ultra-Red portion.*	
K	Calcium	393	A^{IV}	1444

* Tait, *Light*, p. 33.

The rate at which vibrations are received by the eye is the same as the rate at which they are emitted by the luminous body, provided that the distance between the spectator and the luminous body remains constant. A velocity of separation or approach modifies the rate of reception, and consequently the sensation of colour, in the same manner as it modifies the pitch of a musical sound (Art. 235).

REFRACTIVE INDEX OF SOLIDS AND LIQUIDS FOR SODIUM LIGHT (D LINE).

μ ∟ parallel per ∟ along in air = ∟ parallel per ∟ along in substance, about 18° C.

Solid.	μ	*Liquid.*	μ
Canada balsam, .	1·53	Alcohol, ...	1·36
Diamond, . .	2·47	Benzol, ...	1·50
Glass, crown, . .	1·52	Bisulphide of carbon, .	1·63
Glass, flint, . .	1·6	Chloroform, . . .	1·45
Ice, ...	1·31	Ether, ...	1·36
Iceland spar, . .	1·66, 1·49	Oil of cassia, . . .	1·60
Phosphorus, . .	2·14	Oil of turpentine, . .	1·47
Quartz, . . .	1·54, 1·55	Olive oil, ...	1·48
Rocksalt, . . .	1·54	Petroleum, . . .	1·46
Sulphur, . .	1·95, 2·04, 2·24	Water, ...	1·33

ABSOLUTE REFRACTIVE INDEX OF GASES FOR SODIUM LIGHT.

n ∟ parallel per ∟ along in vacuum = ∟ parallel per ∟ along in gas at 0° C. and 760 mm.

Gas.	n	*Gas.*	n
Air	1·000294	Oxygen, . . .	1·000272
Hydrogen, . . .	1·000138	Carbonic acid, . .	1·000449
Nitrogen, . . .	1·000300	Nitrous oxide, . .	1·000503

ART. 249.—**Refractive Index.** When a ray of light passes

from one medium A into another medium B, it is bent out of its former straight course in such a manner that the ratio of the sine of the angle of incidence to the sine of the angle of refraction is constant. This physical ratio is expressed by

μ ∟ parallel per ∟ along in A = ∟ parallel per ∟ along in B.
By "parallel" is meant parallel to the surface of separation of the medium.

This ratio, of which μ denotes the value, is called the refractive index of the medium B relatively to the medium A. The values are commonly given in terms of air, as a standard incident medium. When the incident medium is vacuum, or, to speak more properly, the pure ether, the index is called the absolute index for the medium B.

Refractive Index and Dispersive Power of Water and Glass.

$\frac{\mu_G - \mu_C}{\mu_E - 1}$ degrees dispersion = degree of refraction.

Kind of Light.	Water at 10° C.	Crown Glass (hard).	Flint Glass, extra dense.
A	1·329	1·512	1·639
B	1·330	1·514	1·643
C	1·331	1·515	1·645
D	1·333	1·517	1·650
E	1·335	1·520	1·658
F	1·337	1·523	1·664
G	1·341	1·528	1·677
H	1·344	1·533	1·689
Dispersive power, $\frac{\mu_G - \mu_C}{\mu_E - 1}$	·030	·025	·049

Art. 250.—**Dispersion ; Dispersive Power.** The value of the refractive index is different for the different kinds of light, being greater the smaller the wave-length. The angle between

the extreme red and the extreme violet rays is called the *dispersion.*

By the *dispersive power* of a substance is meant the ratio of the dispersion produced to the angle by which the mean visible ray has been refracted; hence it is expressed in the form

k degs. dispersion of extreme rays = deg. refraction of mean ray.

The rays corresponding to the Fraunhofer lines C and G may be taken for definite extreme rays, and that corresponding to E for the mean ray. Let μ_C, μ_G, μ_E, denote the respective indices of refraction of the substance for the rays mentioned. Then the dispersive power of the substance is

$$\frac{\mu_G - \mu_C}{\mu_E - 1} \text{ degrees dispersion of } G \text{ from } C = \text{degree refraction of } E.$$

ART. 251.—**Rotary Power; Apparent Rotary Power.** When a beam of polarized light is passed through a plate of quartz, the plane of polarization after emergence is found to be turned in the one or the other direction round. The amount of the change is proportional to the thickness of the plate. Hence we have a constant called the rotary power of a substance,

α degrees rotation = **L** thickness of substance.

When the active substance is a liquid, or is dissolved in an inactive liquid, it is more convenient to consider the *apparent rotary power*, which is expressed by

$[\alpha]$ degrees rotation = **L** length of liquid × **M** of active substance per $\mathbf{L}^3$ of liquid.

ROTARY POWER OF A CRYSTAL FOR SODIUM LIGHT.

α degrees rotation = mm. thickness of crystal.

Crystal.	α	*Crystal.*	α
Quartz, ...	± 21·7	Hyposulphate of lead, .	5·5
Sodium chlorate, . .	3·2	Potassium hyposulphate, .	8·4

APPARENT ROTARY POWER FOR SODIUM LIGHT.

$[\alpha]$ degrees rotation = dm. length of liquid × gm. of active substance per c.c. of liquid.

Active Substance.	$[\alpha]$	*Active Substance.*	$[\alpha]$
Cane-sugar, . . .	+ 67	Cinchonine, . . .	+ 235
Milk-sugar, . . .	+ 54	Tartaric acid, . .	+ 15
Lactose, ...	+ 80	Oil of turpentine, . .	+ 14
Grape-sugar, . . .	+ 53	,, . .	− 37
Camphor, . . .	+ 55	Quinine sulphate, . .	− 164

EXAMPLES.

Ex. 1. A lamp and a candle, placed at 10 feet and 3 feet distance respectively from a screen, illuminate it to an equal extent. Compare the illuminating power of the lamp with that of the candle.

The currents of light from the candle and from the lamp are equal at the screen. Take that current as unit, and denote it by **W** per **T** per square foot. The intensity of the candle at unit distance then is

1 **W** per **T** = square foot by $\frac{1}{9}$ (foot distance)^2,

i.e., 9 **W** per **T** = square foot by (foot distance)^2,

i.e., 9 **W** per **T** = steradian.

Similarly, the intensity of the lamp is

100 **W** per **T** per steradian.

Hence the intensity of the lamp is 100/9 of that of the candle; that is, 11 nearly.

Ex. 2. Find the number of vibrations per second made by sodium light (D line), and the period of vibration of a sodium atom.

The wave-length is 590 millionths of mm. = vibration,

i.e., 59×10^{-8} metres = vibration.

The velocity of light is 300×10^6 metres = second.

i.e., 3×10^8 metres = second.

Therefore the frequency is

3×10^8 vibrations = 59×10^{-8} seconds,

i.e., $\frac{3}{59} \times 10^{16}$ vibrations = second,

i.e., $5{\cdot}085 \times 10^{14}$ vibrations = second.

Hence the reciprocal

$1{\cdot}97 \times 10^{-15}$ second = vibration

is the period of vibration of a sodium atom.

Ex. 3. The refractive indices of water and glass relatively to air are 1·33 and 1·52; what is the refractive index of glass relatively to water.

1·33 L parallel in air = L parallel in water,

1·52 L parallel in air = L parallel in glass;

multiply cross-wise, leaving out the unit common to both sides,

1·52 L parallel in water = 1·33 L parallel in glass,

i.e., 1·14 L parallel in water = L parallel in glass.

Ex. 4. Find the rotation produced by a column of solution of cane-sugar, 20 cm. in length, having a strength of 15 gm. per 100 cubic cm.

67 degrees = 10 cm. length by (gm. per c.c.), .

20 cm. length by 15 gm. per 100 c.c.,

$$\therefore \quad \frac{67 \times 20 \times 15}{10 \times 100} \text{ degrees,}$$

i.e., 20·1 degrees.

EXERCISE LII.

1. A gas jet 16 feet and a candle 4 feet from a photometer are found to illuminate it equally. Compare the quantities of light emitted from the two sources.

2. Two gas jets with 6-ft. and 8-ft. burners, when placed at distances of 8 feet and 6 feet from a screen, produce equal illumination; compare their illuminating powers.

3. A standard candle is placed one yard behind a paper screen having a grease spot; at what distance in front must a Carcel burner be placed to neutralize the illuminating effect of the candle?

4. A pencil is held before a paper screen which is simultaneously illuminated by a standard candle and a gas-burner. The standard candle is at a distance of 18 inches, and it is found that the burner is at a distance of 40 inches, when the shadows cast on the screen by the pencil are equally dark. Find the illuminating power of the burner.

5. Express the velocity of light in terms of the earth-quadrant and second.

6. Assuming the velocity of light to be 200,000 miles per second, and the wave-length for a green colour to be the 40,000th part of an inch, how many vibrations will it make per second?

7. The wheel constructed by Fizeau to determine the velocity of light had 720 teeth and the width of a notch was equal to the width of a tooth. An angular speed of 12·6 revolutions per sec. stopped completely on its return a ray which had passed through the notch and been reflected back in the same line by a mirror at the distance of 8,663 metres. Calculate the velocity of light.

8. If the refractive index of a ray of light in passing from air to water be 4/3, and in passing from air to glass 3/2; find what it is for passing from water to glass.

9. The angle of incidence being 60°, and the index of refraction being $\sqrt{3}$, find the angle of refraction.

10. A ray of light falls at an angle of 45° upon the separating plane between air and glass. Determine its path, *first*, when the incident ray is in the air; and *secondly*, when it is in the glass.

11. Find the dispersive power of water, flint-glass, and crown-glass, taking the rays A, H, and D.

12. The indices of refraction of the extreme and mean rays being assumed as 1·5466, 1·5258, 1·5330 for crown-glass, and as 2·4670, 2·4110, 2·4390 for diamond, compare the refracting powers and the dispersive powers of the substances.

SECTION LIII.—GEOMETRICAL.

ART. 252.—**Plane Mirror.** When the light from a luminous point is reflected by a plane mirror, the image is in the same perpendicular with the luminous point, and at a distance behind the mirror equal to the distance of the point before the mirror. Hence the dimensions of the image of an object are given by

1 L distance in image = L corresponding distance in object.

ART. 253.—**Spherical Mirror.** Let the radius of the mirror be denoted by r L. When a luminous point is at the distance u L along the axis from the mirror, the distance of its image

from the mirror along the axis is v L, v being such that

$$\frac{1}{u}+\frac{1}{v}=\frac{2}{r}.$$

The unit which is implied in this equation is evidently the unit of curvature (Art. 74). The full meaning of the equation is that the arithmetical mean of the curvatures at the mirror due to the distances of the point and its image is equal to the curvature of the mirror.

When the source of light is at an infinite distance from the mirror along the axis, $1/u=0$, and therefore $v=r/2$. This distance of the image is called the *focal length.*

Let d L be the distance of an object from the centre of the mirror, and d' L the distance of its image, then

d'/d L length in image = L corresponding length in object,

i.e., $\left(v-\frac{r}{2}\right)\Big/\frac{r}{2}$ " = " "

i.e., $\frac{r}{2}\Big/\left(u-\frac{r}{2}\right)$ " = " "

ART. 254.—**Refraction at a Plane Surface.** When a luminous point inside a refracting medium is looked at perpendicularly to the surface, the image is in the same perpendicular with the luminous point, and its distance is given by the ratio

$1/\mu$ L behind = L behind of point.

Hence the image of an object of moderate size is altered in the perpendicular direction according to the above ratio, and is unaltered in the two transverse directions.

ART. 255.—**Lens.** Let a double-convex spherical lens have radii of curvature r L and r' L, and let the refractive index of the glass relatively to the air be μ. When a luminous point is at a distance u L along the axis from the central point of the lens, its image is in the same axis at a distance v L, such that

$$\frac{1}{u}+\frac{1}{v}=(\mu-1)\left(\frac{1}{r}+\frac{1}{r'}\right);$$

$$= (\mu - 1)\frac{2}{r},$$

if $r' = r$.

The focal length is the reciprocal of the constant value on the right hand; it is denoted by the single letter f.

The dimensions of the image are given by the scale

v/u L distance in image = L corresponding distance in object,

or $(v-f)/f$ „ „ = „ „

or $f/(u-f)$ „ „ = „ „

Examples.

Ex. 1. An object, 6 inches long, is placed symmetrically on the axis of a convex spherical mirror, and at a distance of 12 inches from it; the image formed is found to be 2 inches long. What is the focal length of the mirror?

By Art. 253.
$$\frac{f}{12-f} = \frac{2}{6},$$
$$\therefore \quad f = 3. \qquad \textit{Ans.}\text{—3 inches.}$$

Ex. 2. The focal length of a lens in air is 5 feet. The refractive index of glass and water being 3/2 and 4/3 respectively with respect to air, find the focal length of the lens when placed in water.
$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right),$$
$$\therefore \quad \tfrac{1}{5} = (\tfrac{3}{2} - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right),$$
$$\therefore \quad \frac{1}{r_1} + \frac{1}{r_2} = \tfrac{2}{5}.$$

Now the refractive index of glass relatively to water is (Ex. 3, above) $\frac{3}{4} \times \frac{3}{2}$; hence
$$\frac{1}{f'} = (\tfrac{9}{8} - 1)\,\tfrac{2}{5},$$
$$= \tfrac{2}{40},$$
$$\therefore \quad f' = 20. \qquad \text{Hence 20 feet.}$$

EXERCISE LIII.

1. A candle-flame is placed at a distance of 3 feet from a concave mirror formed of a portion of a sphere, the diameter of which is 3 feet. Determine the position of the image of the candle-flame produced by the mirror, and state whether it is erect or inverted.

2. The radius of a convex mirror is 6 inches. If the linear dimensions of an object be twice those of its image, where must each be situated?

3. A straight stick is partially immersed in water. The image, when looked at normally, is inclined at 45° to the surface, and the index of refraction is 4/3. Find the inclination of the stick.

4. A bright point, 6 inches above the surface of still water, is reflected from the bottom of the vessel, which is 2 feet deep, as well as from the surface of the water. Find the positions of the images formed by the reflections.

5. The eyes of an angler in air and of a salmon in water being supposed each 12 feet from the separating plane of their media; if those of the salmon appear to the angler but 9 feet below it, how far above it will those of the angler appear to the salmon?

6. A convex lens, held 12 inches from the wall, forms on the wall a distinct image of a candle; when the lens is held 6 inches from the wall, it is found that the distance of the candle from the lens must be doubled to produce a distinct image on the wall. Find the focal length of the lens.

7. On a double convex lens, of 10 inches focal length, light is incident from a point 6 inches from the lens. Find the position of the focus.

8. A small object 0·1 inch long is placed at a distance of 3 feet from a convex glass lens of 12 inches focal length. What is the length of its image, and its distance from the lens?

9. The focal length of a convex lens is 1 foot. Determine the position and the character of the image of a small object in the axis of the lens at distances respectively of 20 feet, 2 feet, 10 inches, and 1 inch. Determine also in each case the size of the image compared with that of the object.

10. The chief focal length of a lens is 12 inches; how far must I place a luminous object from the lens in order to obtain an image twice as large every way as the object?

11. Find the geometrical focus of a small pencil of rays refracted through a double convex lens, the radii of which are 9 and 18 inches, and the refractive index 3/2, the origin of light being at a distance of 12 inches.

CHAPTER NINTH.

CHEMICAL.

SECTION LIV.—COMPOSITION.

ART. 256.—**Composition of a Substance.** Suppose that a substance is formed by the combination of three elementary substances, A, B, and C, and that a **M** of A, b **M** of B, and c **M** of C have been used up in forming the substance. A substance is perfectly homogeneous throughout; hence, as in the case of a homogeneous mixture (Art. 25) but with much greater exactness,

$$a\,\mathsf{M} \text{ of } A + b\,\mathsf{M} \text{ of } B + c\,\mathsf{M} \text{ of } C = a + b + c\,\mathsf{M} \text{ of substance.}$$

Partial equivalences and equivalences of the tangent kind are derived as in the case of mixture. Thus the composition is fully given by taking together the equivalences

$$b/a\,\mathsf{M} \text{ of } B = \mathsf{M} \text{ of } A\text{; and } c/a\,\mathsf{M} \text{ of } C = \mathsf{M} \text{ of } A.$$

From these two ratios the third

$$c/b\,\mathsf{M} \text{ of } C = \mathsf{M} \text{ of } B$$

follows by necessary consequence.

These ratios differ from the corresponding ratios of a mixture in the respect that they are each limited to a few values.

ART. 257.—**Combining Weight.** Suppose that a **M** of A and b **M** of B unite to form a substance X. The composition then is

$$a\,\mathsf{M} \text{ of } A + b\,\mathsf{M} \text{ of } B = a + b\,\mathsf{M} \text{ of } X,$$

and the combining weight (by which is meant the combining mass) of B relatively to A for the substance X is

$$b/a \text{ M of } B = \text{M of } A.$$

But b' M of B may combine with a M of A to form another substance Y, so that

$$a \text{ M of } A + b' \text{ M of } B = a + b' \text{ M of } Y;$$

and the combining weight of B relatively to A is in this case

$$b'/a \text{ M of } B = \text{M of } A.$$

It is found that the value b'/a is either a multiple of b/a, or is related to b/a by a simple fraction.

For example, the ratio of oxygen to hydrogen in water is

$$8 \text{ M of oxygen} = \text{M of hydrogen};$$

while in the case of peroxide of hydrogen it is

$$16 \text{ M of oxygen} = \text{M of hydrogen}.$$

By comparing the several elements directly or indirectly with one element, a set of combining weights can be formed. The standard element is hydrogen, and the set of combining weights which has been chosen is exhibited in the following table. The values have been deduced from a comparison of those published by Meyer and Seubert with those published by Clarke.

COMBINING WEIGHTS OF THE ELEMENTS.

r M of element = M of Hydrogen.

Notation.	Name.	r	Notation.	Name.	r
Ag	Silver (*Argentum*),	107·66	*C*	Carbon,	11·97
Al	Aluminium,	27	*Ca*	Calcium,	40
As	Arsenic,	74·9	*Cd*	Cadmium,	112
Au	Gold (*Aurum*),	196·2	*Ce*	Cerium,	141
B	Boron,	10·9	*Cl*	Chlorine,	35·37
Ba	Barium.	136·8	*Co*	Cobalt,	58·6
Be	Beryllium,	9·1	*Cr*	Chromium,	52
Bi	Bismuth,	208	*Cs*	Caesium,	133
Br	Bromine,	79·75	*Cu*	Copper (*Cuprum*),	63·2

COMBINING WEIGHTS OF THE ELEMENTS.—*Continued.*

Notation.	Name.	r	Notation.	Name.	r
D	Didymium, . .	145	*Pd*	Palladium, . .	106
E	Erbium, . . .	166	*Pt*	Platinum, . .	194·4
F	Fluorine, . .	19	*Rb*	Rubidium, . .	85·2
Fe	Iron (*Ferrum*),. .	55·9	*Rh*	Rhodium, . .	104
G	Gallium, . .	69	*Ru*	Ruthenium, . .	103·5
H	Hydrogen, . .	1	*S*	Sulphur, . .	31·98
Hg	Mercury(*Hydrargyrum*)	199·8	*Sb*	Antimony (*Stibium*),	120
I	Iodine, . . .	126·55	*Sc*	Scandium, . .	44
In	Indium, . . .	113·4	*Se*	Selenium, . .	78·8
Ir	Iridium, . . .	193	*Si*	Silicon, . . .	28
K	Potassium (*Kalium*),	39·03	*Sn*	Tin (*Stannum*), .	118
La	Lanthanum, . .	139	*Sr*	Strontium, . .	87·3
Li	Lithium, . .	7·01	*Ta*	Tantalum, . .	182
Mg	Magnesium, . .	24	*Te*	Tellurium, . .	128
Mn	Manganese, . .	54	*Th*	Thorium, . .	232
Mo	Molybdenum, . .	95·8	*Ti*	Titanium, . .	50
N	Nitrogen, . .	14·01	*Tl*	Thallium, . .	204
Na	Sodium (*Natrium*), .	23·00	*U*	Uranium, . .	240
Nb	Niobium, . .	94	*V*	Vanadium, . .	51·2
Ni	Nickel, . . .	58	*W*	Tungsten, . .	184
O	Oxygen, . . .	15·96	*Y*	Yttrium, . .	90
Os	Osmium, . . .	197	*Yb*	Ytterbium, . .	173
P	Phosphorus, . .	30·96	*Zn*	Zinc, . . .	64·9
Pb	Lead (*Plumbum*), .	206·4	*Zr*	Zirconium, . .	90

ART. 258.—**Empirical Formula.** The symbols *Ag*, *Al*, etc., may be used in a qualitative sense merely, or they may have a quantitative signification attached. In the latter case the quantity attached to the qualitative meaning of the symbol is the value of the combining weight.

If the combining weights of *A*, *B*, *C*, the elements forming a compound are α, β, γ relatively to hydrogen, and the multiples according to which these combining weights occur in the par-

ticular substance p, q, r; then the composition of the substance is

$p\alpha$ **M** of $A + q\beta$ **M** of $B + r\gamma$ **M** of $C = p\alpha + q\beta + r\gamma$ **M** of compd.;

and an empirical formula for the compound is derived by a short notation for this equivalence, namely, $A_p\ B_q\ C_r$ reduced to its simplest form.

The notation $A_p\ B_q\ C_r$ in its simplest form expresses the combining weight of the compound, and it also serves for a name, excepting in the case of the class of substances which have the same chemical composition, but different physical properties.

ART. 259.—**Percentage Composition.** The composition of a substance $A_p\ B_q\ C_r$ is

$p\alpha$ **M** of $A + q\beta$ **M** of $B + r\gamma$ **M** of $C = p\alpha + q\beta + r\gamma$ **M** of subst.

This equivalence will remain true, after each term is multiplied by 100 and divided by $p\alpha + q\beta + r\gamma$. Hence

$$\frac{100\,p\alpha}{p\alpha + q\beta + r\gamma}\ \mathbf{M} \text{ of } A + \frac{100\,q\beta}{p\alpha + q\beta + r\gamma}\ \mathbf{M} \text{ of } B + \frac{100\,r\gamma}{p\alpha + q\beta + r\gamma}\ \mathbf{M} \text{ of } C$$
$$= 100\ \mathbf{M} \text{ of substance.}$$

Let the values of the three percentages be respectively n_1, n_2, n_3. Then we deduce the equations

$$\frac{q\beta}{p\alpha} = \frac{n_2}{n_1}, \quad \frac{r\gamma}{p\alpha} = \frac{n_3}{n_1}.$$

There are only two independent equations. Hence from the percentage data given by analysis we can determine only the ratios to one of the elements of each of the remaining elements, that is, the empirical formula. To determine the *molecular* formula an additional datum is required (Art. 266).

ART. 260.—**Composition by Volume.** Suppose that the three elements A, B, C are gases, having at a common temperature and pressure the bulks x, y, z **V** per **M**. Let the composition by mass of a substance X formed by them be

$p\alpha$ **M** of $A + q\beta$ **M** of $B + r\gamma$ **M** of $C = p\alpha + q\beta + r\gamma$ **M** of X.

The composition by volume is obtained by finding the volumes of

the several components. The volume of A is pax, etc.; therefore the composition by volume is

$$pax \ \mathsf{V} \text{ of } A + q\beta y \ \mathsf{V} \text{ of } B + r\gamma z \ \mathsf{V} \text{ of } C = pa + q\beta + r\gamma \ \mathsf{M} \text{ of } X.$$

The mass of a compound is necessarily equal to the sum of the masses of its components; but a similar statement is not true for volume.

Here the composition by volume is deduced from the composition by mass, in the same manner as in the case of mixture of a commodity the composition by value is deduced from the composition by mass (Art. 26).

ART. 261.—**Strength of a Solution; Solubility.** The composition of a solution is expressed in the form

$$q \ \mathsf{M} \text{ of solid} + (1-q) \ \mathsf{M} \text{ of solvent} = \mathsf{M} \text{ of liquid.}$$

The strength of the solution is expressed by

$$q/(1-q) \ \mathsf{M} \text{ of solid dissolved} = \mathsf{M} \text{ of solvent.}$$

The ratio $q/(1-q)$ is not restricted to a few values like a ratio of combination, but may have any fractional value up to a certain limit, which depends on the temperature of the solvent.

By the *solubility* of a solid in a liquid at a given temperature is meant the maximum value of the strength. The reciprocal idea is the minimum value of

$$(1-q)/q \ \mathsf{M} \text{ of solvent} = \mathsf{M} \text{ of solid dissolved;}$$

it gives the idea of resistance to solution.

ART. 262.—**Absorptive-power.** The solution of a gas in a liquid is called absorption. The composition of the solution is expressed by

$$q \ \mathsf{M} \text{ of gas absorbed} + (1-q) \ \mathsf{M} \text{ of solvent} = \mathsf{M} \text{ of liquid,}$$

and the strength of the solution by

$$q/(1-q) \ \mathsf{M} \text{ of gas absorbed} = \mathsf{M} \text{ of solvent.}$$

If the bulk of the gas before absorption is v V per M, then

$$qv/(1-q) \ \mathsf{V} \text{ of gas absorbed} = \mathsf{M} \text{ of solvent;}$$

and if the density of the absorbing liquid is ρ M per V, then

$$qv\rho/(1-q) \ \mathsf{V} \text{ of gas absorbed} = \mathsf{V} \text{ of solvent.}$$

The absorptive power of the substance is the maximum value of any one of these three equivalent rates.

EXAMPLES.

Ex. 1.—Calculate the percentage composition of nitric acid.

The formula is HNO_3; hence the composition is

$$1 \text{ M of } H + 14 \text{ M of } N + 3 \times 16 \text{ M of } O = 1 + 14 + 48 \text{ M of } HNO_3.$$

Hence $$\frac{100}{1+14+48} \text{ M of } H = 100 \text{ M of } HNO_3,$$

i.e., $$1{\cdot}6 \text{ M of } H = 100 \text{ M of } HNO_3,$$

Similarly $$22{\cdot}2 \text{ M of } N = 100 \text{ M of } HNO_3,$$

and $$76{\cdot}2 \text{ M of } O = 100 \text{ M of } HNO_3.$$

Ex. 2.—Dumas found that water contains 88·864 per cent. of oxygen. Deduce the empirical formula for water.

$$88{\cdot}864 \text{ M of } O + 11{\cdot}136 \text{ M of } H = 100 \text{ M of water},$$

$$\therefore \quad \frac{88{\cdot}864}{11{\cdot}136} \text{ M of } O = \text{M of } H;$$

but $$n \times 15{\cdot}96 \text{ M of } O = \text{M of } H,$$

$$\therefore \quad n = \frac{88{\cdot}864}{11{\cdot}136 \times 15{\cdot}96}$$

$$= \tfrac{1}{2}.$$

Hence $$HO_{\frac{1}{2}} \text{ or } H_2O.$$

Ex. 3.—Find the empirical formula of a substance the analysis of which yields the following results—Carbon 20·00, Hydrogen 6·67, Nitrogen 46·67, Oxygen 26·66.

Let it be $C_pN_qO_rH$; then

$$p\ 12 \text{ M of } C = \text{M of } H,$$

$$q\ 14 \text{ M of } N = \text{M of } H,$$

$$r\ 16 \text{ M of } O = \text{M of } H;$$

hence $$p\ 12 = \frac{2000}{667},$$

$$\therefore \quad p = \frac{500}{2001} = \tfrac{1}{4}.$$

Similarly q is found to be $\frac{1}{2}$, and r to be $\frac{1}{4}$. Hence the empirical formula is $C_{\frac{1}{4}}N_{\frac{1}{2}}O_{\frac{1}{4}}H$ or CN_2OH_4.

Ex. 4.—From the following data calculate the composition of water by weight, the density of oxygen being 15·96.

Volume of oxygen taken	95·45
Volume of oxygen + hydrogen	557·26
Volume after explosion	271·06.

Let **V** denote the unit of volume.

95·45 **V** of O combined,

271·06 **V** of H left,

∴ 366·41 **V** of O combined plus H left;

but 557·26 **V** of O combined plus H combined plus H left,

∴ 190·85 **V** of H combined.

Hence the composition of water is

95·45 **V** of O + 190·85 **V** of H,

i.e., 1 **V** of O + 2 **V** of H.

But 15·96 **M** per **V** of O = **M** per **V** of H,

∴ 15·96 **M** of O + 2 **M** of H,

or 7·98 **M** of O + **M** of H.

EXERCISE LIV.

1. State the composition of ferrocyanide of potassium, and of hyposulphite of soda.
2. How much carbon is there in 100 grammes of marsh gas?
3. Give the percentage composition of iron pyrites and of epsom salt.
4. What is the percentage composition of calomel, corrosive sublimate, oxalic acid?
5. Calculate the percentage composition of mercurous nitrate.
6. As the result of nineteen experiments Dumas found that 840·161 grammes of oxygen were consumed in the production of 945·439 grammes of water. Calculate from that datum the percentage composition of water by weight.
7. The percentage composition of a body is

Carbon	54·55
Hydrogen	9·09
Oxygen	36·36
	100·00

Deduce a formula for the body.

8. A salt has the following percentage composition:—Sodium 22·55; Phosphorus 30·39; Oxygen 47·06. Calculate the simplest formula.

9. A salt on analysis gives the following percentage numbers—$N = 9·09$; $O = 20·77$; $Ag = 70·13$. Calculate the simplest formula.

10. 24·94 grains of an iron ore yielded on analysis—

Copper	8·63	grains.
Iron	7·61	,,
Sulphur	8·70	,,

To what empirical formula does the analysis conduct?

11. 50 cc. of oxygen are mixed with 500 cc. of hydrogen, both measured at the normal temperature and pressure. An electric spark is passed through the mixture, what volume of gas will be left?

12. Water consists of one volume of oxygen and two volumes of hydrogen. Taking the specific gravity of oxygen as 1·108, and that of hydrogen as ·0693, what is the percentage composition of water?

13. Acetic acid is found on combustion to yield 40 per cent. of carbon, 6·6 per cent. of hydrogen, and by difference 53·4 per cent. of oxygen. What conclusion can be drawn from these data?

14. The coefficients of absorption of oxygen and nitrogen being 0·0411 and 0·0203 respectively; find the percentage volume composition of air dissolved in water, on the supposition that air is a mixture, and that these gases obey Dalton's law.

SECTION LV.—ATOM AND MOLECULE.

ART. 263.—**Atomic Weight.** Let us suppose that an elementary substance is made up of atoms which are precisely alike in all respects. Let $\mathbf{M}_H$ denote not any unit of mass, but the mass of an atom of hydrogen. Then from the combining weight of any element we derive the mass of its atom, so that the table of combining weights becomes a table of atomic weights. Chemists have chosen that particular set of combining weights which is believed to be identical with the set of atomic weights. The mass of the atom of silver is 107·66 $\mathbf{M}_H$, of oxygen 15·96 $\mathbf{M}_H$, and so on.

The combining weight of a compound becomes the mass of its compound atom.

ART. 264.—**Atomic Heat.** By Art. 183 the specific heat of a solid substance is expressed by

$$s\ \mathbf{M} \text{ of water} = \mathbf{M} \text{ of substance.}$$

Take as a particular unit of mass M_H, the mass of an atom of hydrogen; then the specific heat of an element when solid is expressed by

$$s\ \mathbf{M}_H \text{ of water} = \mathbf{M}_H \text{ of element.}$$

Now $\qquad r\ \mathbf{M}_H$ of element = atom of element,

therefore $\qquad sr\ \mathbf{M}_H$ of water = atom of element,

This equivalence is called the *atomic heat* of the element; its value is the product of the value of the specific heat and atomic weight of the element. Its value is nearly the same for the different elements, namely about 6·2.

This relation affords one means of determining which of the combining weights of an element is its principal combining weight.

ART. 265.—**Molecular Weight.** By a *molecule* of a substance is meant the group of atoms which exist together, when the substance is in a free state. If the substance is elementary the atoms in a molecule are necessarily of the same kind; if the substance is compound, the molecule may consist of one group of the constituent atoms, as in the case of hydrochloric acid, or of several such groups, as in the case of acetylene.

Let $\mathbf{M}_H$ denote as before the mass of an atom of hydrogen; then the mass of a molecule expressed in terms of $\mathbf{M}_H$ is called the *molecular weight* of the substance.

ART. 266.—**Number of Atoms in a Molecule; Molecular Formula.** If we assume that the molecule of hydrogen contains 2 atoms, then the number of atoms in the molecule of any other gas can be determined by comparing its density with that of hydrogen under the same conditions of temperature and pressure.

Let the atomic weight of the gas be

$$r\ \mathbf{M}_H = \text{atom (or compound atom),}$$

the number of atoms in a molecule

$$n \text{ atoms} = \text{molecule},$$

and the number of molecules in unit of volume

$$q \text{ molecules} = \mathbf{V};$$

then we deduce

$$rnq\ \mathbf{M}_H \text{ of gas} = \mathbf{V}.$$

In the case of hydrogen r is 1 and n is 2, and q is the same for all gases, therefore

$$2q\ \mathbf{M}_H \text{ of hydrogen} = \mathbf{V}.$$

Hence

$$rn\ \mathbf{M}_H \text{ of gas per } \mathbf{V} = 2\ \mathbf{M}_H \text{ of hydrogen per } \mathbf{V}.$$

Now this expresses the relative density; hence if its value is known, say d, we have the equation

$$\frac{rn}{2} = d.$$

The molecular formula expresses not only the atom or compound atom, but also the number of atoms or compound atoms in the molecule.

The density of hydrogen at 0° and 760 mm. is

$$\cdot 0895 \text{ gm. per litre.}$$

MOLECULAR WEIGHTS OF THE GASES.

$m\ \mathbf{M}_H$ per molecule of the gas.

Gas.	Molecular Formula.	m	Gas.	Molecular Formula.	m
Acetylene, - -	C_2H_2	26	Marsh gas, - -	CH_4	16
Aethylene, - -	C_2H_4	28	Nitric oxide, - -	NO	30
Ammonia, - -	NH_3	17	Nitrogen, - -	N_2	28
Bromine, - -	Br_2	159·5	Nitrous oxide, -	N_2O	44
Carbonic acid, -	CO_2	44	Oxygen, - -	O_2	32
Chlorine, - -	Cl_2	70·75	Phosphuretted hyd.	PH_3	34
Cyanogen, - -	C_2N_2	52	Sulphuretted hyd.	H_2S	34
Hydrochloric acid,	HCl	36·375	Sulphurous acid, -	SO_2	64
Hydrogen, - -	H_2	2	Water vapour, -	H_2O	18

Examples.

Ex. 1. The analysis of an acid gave in a hundred parts 48·648 parts of C, 8·108 of H, and 43·243 of O. The molecular weight of the acid is 74 ; calculate its chemical formula.

Let its composition be

$$p\ \mathbf{M} \text{ of } C + q\ \mathbf{M} \text{ of } O + 1\ \mathbf{M} \text{ of } H;$$

but its composition is given as

$$48{\cdot}648\ \mathbf{M} \text{ of } C + 43{\cdot}243\ \mathbf{M} \text{ of } O + 8{\cdot}108\ \mathbf{M} \text{ of } H;$$

therefore $\quad p = \frac{48648}{8108} = 6 = \frac{1}{2} \text{ of } 12,$

and $\quad q = \frac{43243}{8108}.$

Now

```
8108)43243(5
     40540
     -----
      2703)8108(3
           8109
           ----
```

$$\therefore \quad q = 5 + \tfrac{1}{3} = \tfrac{16}{3}.$$

Hence $\quad C_{\frac{1}{2}} H\, O_{\frac{4}{3}}$ or $C_3 H_6 O_2$.

Now, if $C_3 H_6 O_2$ were the molecular formula of the acid, its molecular weight would be

$$3 \times 12 + 6 \times 1 + 2 \times 16\, \mathbf{M}_H,$$

i.e., $\quad 74\ \mathbf{M}_H,$

which is the given value.

Ex. 2. The density of hydrogen is 1, and its molecular weight is 2 ; the density of marsh gas is 8, what is its molecular weight ?

The composition of marsh gas is CH_4; let its molecule be $C_n H_{4n}$. Then

$$n\, 12 + 4n\, 1\ \mathbf{M} \text{ of } CH_4 \text{ per } \mathbf{V} = 2\ \mathbf{M} \text{ of } H \text{ per } \mathbf{V};$$

but $\quad 8\ \mathbf{M}$ of CH_4 per $\mathbf{V} = \mathbf{M}$ of H per $\mathbf{V}$,

therefore $\quad n(12 + 4) = 16,$

i.e., $\quad n = 1.$

Hence the molecular weight of $C H_4$ is 16 $\mathbf{M}_H$.

Ex. 3. What weight of sulphur is contained in 10 litres of sulphuretted hydrogen at 1,000 mm. and 27° C.

32 gm. of S = gm. of H,

1 gm. of $H = 11 \cdot 2$ litres at 0° and 760 mm.,

1 litre at 0° and 760 mm. $= \frac{760}{1000}(1 + \frac{27}{273})$ litres at 27° and 1000 mm.,

10 litres at 27° and 1000 mm.;

$$\therefore \quad \frac{32 \times 10 \times 1000 \times 273}{11 \cdot 2 \times 760 \times 300} \text{ gm. of } S,$$

i.e., 34 gm. of S.

Ex. 4. The atomic heat of potassium is 6·2; what is its specific heat?

$$6 \cdot 2 \ \mathsf{M}_H \text{ of water} = \text{atom of } K,$$

$$39 \ \mathsf{M}_H = \text{atom of } K;$$

$$\therefore \quad \frac{6 \cdot 2}{39} \mathsf{M}_H \text{ of water} = \mathsf{M}_H \text{ of } K,$$

i.e., $\cdot 16 \ \mathsf{M}$ of water $= \mathsf{M}$ of K.

EXERCISE LV.

1. Calculate the density of carbon dioxide from the following data, assuming the temperature and pressure to remain constant.

Weight of globe full of air, - - - - -	948	grammes.
Weight of globe exhausted, - - - - -	933·5	,,
Weight of globe full of carbon dioxide, - -	955·54	,,

2. Find the volume per gramme (1) of oxygen at 15° C. and 770 mm., (2) of nitric oxide at 20° C. and 760 mm.

3. What is the weight of carbon in 1000 litres of carbon dioxide under the standard conditions of temperature and pressure?

4. If 100 cubic inches of ammonia were completely decomposed by the electric spark, what space would its constituents occupy?

5. State the percentage by volume of oxygen and nitrogen in the atmosphere, and from it derive the percentage by weight, assuming the density of each gas to be represented by its atomic weight. Deduce the density of air relatively to hydrogen.

6. 200 grammes of silver afforded 314·894 grammes of the nitrate. Assuming the formula of the salt to be $AgNO_3$, and the atomic weights of silver and oxygen to be 107·93 and 16 respectively, calculate the atomic weight of nitrogen.

7. The densities of hydrogen and marsh gas relatively to air are ·0692 and

·559 respectively. Calculate the molecular weight of marsh gas, assuming that of hydrogen to be 2.

8. The specific gravities of hydrogen and ammonia relatively to air are ·0692 and ·595 respectively. What is the molecular weight of ammonia?

9. A gas of the empirical formula CH_2 is found to possess the density ·978. Calculate its probable molecular weight.

10. According to Dumas' determination of the vapour-density of amylic alcohol, its molecular weight ought to be 90·949. Give the correct molecular weight; and calculate the theoretical and experimental vapour-densities relatively to hydrogen.

11. What is the weight of a litre at 0° and 760 mm. of the vapour of a compound, the normal molecule of which is SO_3.

12. The analysis of an organic silver salt gave 17·34 per cent. of carbon, 1·73 per cent. of hydrogen, 62·44 per cent. of silver, and 18·47 per cent. of oxygen. One molecule of the salt contains two atoms of silver. Calculate its formula.

13. The atomic heat of silver is 6·04, and the atomic weight 108. Find the specific heat of the metal.

14. Find the specific heat of chlorine in the solid state from the following numbers:—

Molecular weight of KCl, - - - - - - -	74·6
Specific heat of KCl, - - - - - - - -	·171
Atomic weight of K, - - - - - - - -	39·1
Specific heat of K, - - - - - - - -	·1655

SECTION LVI.—COMBINATION.

ART. 267.—**Chemical Equation.** As an example of a chemical equation we may take the one which represents the complete combustion of marsh gas—

$$CH_4 + 2O_2 = CO_2 + 2H_2O.$$

As each formula involved is in its molecular form, the equation expresses not only the relative masses but also the relative volumes of the substances, provided the substance is in the form of vapour. It is equivalent in meaning to the two following—

$$1 \text{ V of } CH_4 + 2 \text{ V of } O_2 = 1 \text{ V of } CO_2 + 2 \text{ V of } H_2O;$$

$$16 \text{ M of } CH_4 + 64 \text{ M of } O_2 = 44 \text{ M of } CO_2 + 36 \text{ M of } H_2O,$$

i.e., $4 \text{ M of } CH_4 + 16 \text{ M of } O_2 = 11 \text{ M of } CO_2 + 9 \text{ M of } H_2O.$

From these equations we can derive rates of the form

$$1 \mathsf{V} \text{ of } CO_2 = \mathsf{V} \text{ of } CH_4,$$

$$11 \mathsf{M} \text{ of } CO_2 = 4 \mathsf{M} \text{ of } CH_4.$$

In the case of a complete equation, the value of **M** for any element on the one side must be equal to the value of **M** for the same element on the other side. Hence, if there are n elements involved, we can derive n equations which are sufficient to determine n unknown numbers.

ART. 268.—**Heat of Combination.** When two substances combine chemically, there is simultaneously a transformation of energy previously existing in a potential form. This energy may, by suitable arrangements, be changed into that of an electric current, or it may be changed directly into heat. The amount of heat developed during the production of unit of mass of the substance is called the *heat of combination*, and is expressed in the form

$$k \mathsf{H} \text{ developed} = \mathsf{M} \text{ of substance formed.}$$

ART. 269.—**Calorific Power; Calorific Intensity.** When the process of combination is ordinary combustion, the heat of combination is measured with reference to the substance burned. It is called the calorific power of the combustible, and is expressed in the form

$$k \mathsf{H} = \mathsf{M} \text{ of combustible substance.}$$

By the calorific intensity is meant the number of units by which the temperature of the product is raised above that of the elements before combustion.

The calorific power of fuel is sometimes expressed by the number of pounds of water which can be raised from a given temperature to 100° C. and evaporated under the pressure of one atmosphere by the combustion of one pound of the fuel. This rate is called the evaporating power of the fuel.

HEAT OF COMBINATION WITH OXYGEN.

k kgm. of water by deg. Cent. = kgm. of combustible.

COMBUSTIBLE.	PRODUCT.	$k/10^2$	COMBUSTIBLE.	$k/10^2$
Carbon,	CO_2	79	Marsh gas,	130
Wood-charcoal,		81	Alcohol,	69
Gas-coke,		80	Wood,	27 to 29
Graphite,		78	Coal,	64 to 83
Diamond,		78	Coal gas,	106
Hydrogen,	H_2O	346	Petroleum,	114
Sulphur,	SO_2	22	Turpentine,	107

EXAMPLES.

Ex 1. If air contains 23 per cent. of its weight of oxygen, how many pounds of carbon must be burnt in order to remove all the oxygen from 500 pounds of air?

$$500 \text{ lb. of air}$$
$$23 \text{ lb. of } O = 100 \text{ lb. of air,}$$
$$12 \text{ lb. of } C = 2 \times 16 \text{ lb. of } O;$$
$$\therefore \quad \frac{5 \times 23 \times 12}{2 \times 16} \text{ lb. of } C.$$

i.e., 43 lb. of carbon.

Ex. 2. How many grammes of oxygen are needed to burn completely 100 grammes of bisulphide of carbon.

The process of combustion of the bisulphide is given by the chemical equation

$$CS_2 + 3O_2 = CO_2 + 2SO_2.$$

Hence $\quad 3 \times 2 \times 16$ gm. of $O = 12 + 2 \times 32$ gm. of CS_2,

$$100 \text{ gm. of } CS_2;$$
$$\therefore \quad \frac{100 \times 6 \times 4}{3 + 16} \text{ gm. of } O,$$

i.e., 126 gm. of oxygen.

Ex. 3. One hundred volumes of a hydrocarbon at 0° and 760 mm. require for their complete combustion 200 volumes of oxygen, and furnish 100 volumes of carbonic acid. Find the formula of the hydrocarbon.

Let it be C_nH_m; then

$$1 \text{ V of } C_nH_m + 2 \text{ V of } O_2 = 1 \text{ V of } CO_2 + x \text{ M of } H_2O.$$

As the mass of each element is the same after as before, we deduce

$$n = 1,\ m = 2x,\ 4 = 2 + x;$$

from which $$m = 4.$$

Hence formula is CH_4.

Ex. 4. 0·31 grammes of an organic acid yields on combustion 0·62 grammes of carbonic acid and 0·2536 grammes of water. Find the formula of the acid, and name an acid having the formula.

Let the empirical formula be C_pH_qO.

It is given that

31 gm. of $C_pH_qO + x$ gm. of $O = 62$ gm. of $CO_2 + 25·36$ gm. of H_2O.

In C_pH_qO we have

$$\frac{p12}{p12 + q + 16} \text{ gm. of } C = \text{gm. of } C_pH_qO,$$

therefore the quantity of C is

$$\frac{31p12}{p12 + q + 16} \text{ gm.}$$

But the quantity of C is also

$$\frac{62 \times 12}{12 + 32} \text{ gm.}$$

Hence $$\frac{p}{p12 + q + 16} = \frac{1}{22},$$

from which $$p10 = q + 16. \qquad (1)$$

By equating the quantities of H, we obtain in a similar manner

$$2p = q. \qquad (2)$$

From these equations we obtain $p = 2$, and $q = 4$; hence the formula is C_2H_4O.

Acetyl aldehyde has that composition.

Ex. 5. What volume of oxygen, measured at 27° and 740 mm., can be obtained from 54 grammes of mercuric oxide.

The mode in which mercuric oxide decomposes at the temperature mentioned is stated by the equation

$$HgO = Hg + O,$$

which means

$$216 \text{ gm. of } HgO = 200 \text{ gm. of } Hg + 16 \text{ gm. of } O.$$

Hence $\qquad 2 \text{ gm. of } O = 27 \text{ gm. of } HgO,$

$$54 \text{ gm. of } HgO;$$

$$\therefore \quad 4 \text{ gm. of oxygen.}$$

Now $\qquad 1 \text{ gm. of } H = 16 \text{ gm. of } O,$

$$11{\cdot}2 \text{ litres of } H \text{ at } 0^\circ \text{ and } 760 \text{ mm.} = \text{gm. of } H,$$

$$\tfrac{760}{740}(1 + \tfrac{27}{273}) \text{ litres at } 27 \text{ and } 740 = \text{litre at } O \text{ and } 760;$$

$$\therefore \quad \frac{760 \times 300 \times 11{\cdot}2 \times 4}{740 \times 273 \times 16} \text{ litres of } O,$$

i.e., $\qquad 3{\cdot}2$ litres of oxygen.

EXERCISE LVI.

1. The combustion of hydric sulphide in air is represented by the equation

$$H_2S + O_3 = H_2O + SO_2;$$

how much sulphurous acid and how much water can be formed from 100 grammes of hydric sulphide?

2. How many grammes of phosphorus will be required, when completely burnt, to take the whole of the oxygen out of one kilogramme of air?

3. How many litres of oxygen in the standard state are required to burn completely, *first,* 120 grammes of sulphur, *second,* 155 grammes of phosphorus?

4. How many litres of carbon dioxide would be produced by the complete combustion of 100 grammes of marsh gas?

5. What volume of oxygen, measured at 15° C. and 1000 mm. pressure, is required for the complete combustion of ten grammes of phosphorus?

6. How much phosphuretted hydrogen, by weight and by volume, could be furnished by 31 grammes of phosphorus, supposing all the phosphorus to be converted into the gas?

7. How many volumes of oxygen are needed to burn completely 20 volumes of olefiant gas, and what is the volume of carbon dioxide produced?

8. If two volumes of marsh gas, weighing 16 grammes, be burnt in excess

of air, how many volumes of carbon dioxide are produced; and what is the weight of the other compound formed at the same time?

9. A mixture of 200 volumes of marsh gas and 400 volumes of oxygen measured at 300° C. is collected over dry mercury. After an electric spark has passed through the mixture, what total volume and what gases will remain, the temperature being unchanged?

10. If a pound of carbon be burned in air at the pressure of one atmosphere, and the temperature 62° F., calculate the mass and volume of the air required for complete combustion.

11. How many litres of dry atmospheric air, at 740 mm. and 15° C., are required to burn completely one litre of olefiant gas at the same temperature and pressure. What is the weight of the carbonic anhydride and of the water produced by the combustion?

12. What amount of air is required to burn completely 3·25 cubic feet of olefiant gas?

13. A quantity of sulphur, weighing 4 grammes, is burned in a close glass vessel containing 28·5 litres of pure dry air at 27° and 760 mm. What is the volume of the resulting gaseous mixture at 0° and 760 mm., and what is its composition per cent.?

14. One volume of a gaseous hydrocarbon, measured at 100°, yields on combustion double its volume of carbon dioxide, and three times its volume of steam at the above temperature. Required the formula and the molecular weight of the gas.

15. If one litre of olefiant gas were completely broken up by chlorine, how much hydrochloric acid would be produced, and what weight of carbon would be deposited?

16. What volume of oxygen can be obtained by heating 37·3 grammes of black oxide of manganese, *first* alone, *second* with strong sulphuric acid?

17. The combustion of one cwt. of coal is able to convert 84 gallons of water at 68° into steam at 250°; what is the calorific power of the fuel?

18. A specimen of coal contains 80 per cent. of carbon and 4 per cent of hydrogen uncombined with oxygen. How many gramme-degrees of heat are generated by the combustion of one gramme of this coal?

19. The area of the piston of a high-pressure engine, having a single cylinder, is 200 square inches, and the length of the stroke is 3 feet. If the average pressure behind the piston exceed that in front of it by 25 pound-weight per square inch, and if the fly-wheel make 100 revolutions per minute, what is the horse-power of the engine?

20. If the above engine consume 300 pounds of coal per hour, what percentage of the heat generated in the furnace is converted into useful work? Take 6,400 as the calorific power of the coal.

SECTION LVII.—EQUIVALENCE.

ART. 270.—**Acid and Base.** The equivalence between an acid and a base has the form

$$k \text{ M of acid} = \text{M of base},$$

and it connects quantities of matter having equal but opposite powers of changing the colour of test paper.

$$\text{Suppose } k_1 \text{ M of acid } A = \text{M of base } X,$$

$$\text{and } k_2 \text{ M of acid } B = \text{M of base } X,$$

then we deduce

$$\{k_2/k_1 \text{ M of acid } B = \text{M of acid } A\} \text{per M of } X.$$

The quantities connected have equal acidity with respect to the base X.

Similarly, if

$$q_1 \text{ M of base } X = \text{M of acid } A,$$

$$\text{and } q_2 \text{ M of base } Y = \text{M of acid } A;$$

$$\text{then } \{q_2/q_1 \text{ M of base } Y = \text{M of base } X\} \text{per M of } A.$$

Here the quantities connected have equal basic power with respect to the acid A.

It is found that k_2/k_1 is the same for all bases, and q_2/q_1 the same for all acids.

ART. 271.—**Salt.** Most salts can be viewed as composed of an acid and a base. Take, for example, sulphate of potash. Its composition can be expressed by K_2O,SO_3, which means, when fully expressed,

$$94 \text{ M of } K_2O + 80 \text{ M of } SO_3 = 174 \text{ M of } K_2SO_4.$$

Now, it is found that the value of the ratio of combination

$$47/40 \text{ M of } K_2O = \text{M of } SO_3$$

is either identical with, or bears a simple relation to, the value of the acid-equivalence between K_2O and SO_3.

ART. 272.—**Metal and Salt-radical.** A salt may also be viewed as a compound of a metal and a salt-radical. Thus the composition of cupric sulphate may be expressed by

$$63 \text{ M of } Cu + 96 \text{ M of } SO_4 = 159 \text{ M of } CuSO_4.$$

The composition of hydric sulphate is

$$2 \text{ M of } H + 96 \text{ M of } SO_4 = 98 \text{ M of } H_2SO_4.$$

Hence we deduce

$$63 \text{ M of } Cu \text{ per M of } SO_4 = 2 \text{ M of } H \text{ per M of } SO_4.$$

or $$\{36{\cdot}5 \text{ M of } Cu = \text{M of } H\} \text{per M of } SO_4.$$

This rate expresses the number of mass-units of Cu which replace one mass-unit of hydrogen in the sulphate. It is found that the number for any other salt is either 36·5 or a multiple of it. Hence the general truth may be expressed by

$$\{n\ 36{\cdot}5 \text{ M of } Cu = \text{M of } H\} \text{per M of salt-radical.}$$

By the "equivalent" of a salt-radical is meant the ratio of combination of the salt-radical with hydrogen as

$$48 \text{ M of } SO_4 = \text{M of } H.$$

ART. 273.—**Rate of Electrolysis.** When a salt, dissolved in water, is placed in a voltaic circuit, it is decomposed by the current primarily into metal and salt-radical. The rate of deposition of the metal is proportional to the current; hence we have a rate of the form

$$k \text{ M of metal per T} = \text{ampere.}$$

Its actual value in the case of hydrogen is

$$10415 \times 10^{-9} \text{ gm. per sec.} = \text{ampere.}$$

Let two salts, A and B, be placed in series in the circuit, and let their rates of electrolysis be respectively

$$k_1 \text{ M of } A \text{ metal per T} = \text{ampere,}$$

and $$k_2 \text{ M of } B \text{ metal per T} = \text{ampere.}$$

Now, the strength of the current is the same at each cross-section of the circuit, and the time is the same, therefore

$$\{k_2/k_1 \text{ M of } B \text{ metal} = \text{M of } A \text{ metal}\} \text{ per T.}$$

It is found that k_2/k_1 is identical with the value of the rate of replacement of the metals.

EXAMPLES.

Ex. 1. The equivalents of lithium and of sodium being 7 and

23, find what volume of hydrogen gas at 27° and 1 metre will be produced by the action of 100 grammes of each metal upon water.

$$100 \text{ gm. of } Li$$
$$1 \text{ gm. of } H = 7 \text{ gm. of } Li,$$
$$11{\cdot}2 \text{ litres at } 0° \text{ and } 760 \text{ mm.} = \text{gm. of } H,$$
$$\frac{760}{1000}\,\frac{273+27}{273} \text{ litres at } 27° \text{ and } 1000 \text{ mm.} = \text{litre at } 0° \text{ and } 760 \text{ mm.};$$
$$\therefore \quad \frac{100 \times 11{\cdot}2 \times 760 \times 300}{7 \times 1000 \times 273} \text{ litres,}$$
$$i.e., \quad 134 \quad ,,$$

Second part.

$$100 \text{ gm. of } Na$$
$$7 \text{ gm. of } Li = 23 \text{ gm. of } Na,$$
$$134 \text{ litres of } H = 100 \text{ gm. of } Li;$$
$$\therefore \quad \frac{134 \times 7}{23} \text{ litres of } H,$$
$$i.e., \quad 41 \quad ,,$$

Ex. 2. What weight of silver will be precipitated as chloride, if 22·38 litres of hydrochloric acid are absorbed by a solution of argentic nitrate?

$$AgNO_3 + HCl = AgCl + HNO_3,$$
$$\therefore \quad 108 \text{ gm. of } Ag = 36{\cdot}5 \text{ gm. of } HCl,$$
$$36{\cdot}5 \text{ gm. of } HCl \text{ per litre} = 2 \text{ gm. of } H \text{ per litre,}$$
$${\cdot}0896 \text{ gm. of } H \text{ per litre;}$$
$$\therefore \quad \frac{108 \times {\cdot}0896}{2} \text{ gm. of } Ag \text{ per litre of } HCl,$$
$$22{\cdot}38 \text{ litres of } HCl;$$
$$\therefore \quad 54 \text{ gm. of silver.}$$

Ex. 3. A current of 10 amperes is passed for 20 minutes through solutions of cuprous chloride and of cupric chloride, placed in series; how many grammes of copper will be precipitated in each solution?

In the case of the cuprous chloride,

$$104 \times 10^{-7} \text{ gm. of } H \text{ per sec.} = \text{ampere,}$$

$$63 \text{ gm. of } Cu = \text{gm. of } H,$$
$$10 \text{ amperes};$$
$$\therefore \quad 104 \times 63 \times 10^{-6} \text{ gm. of } Cu = \text{sec.}$$
$$600 \text{ sec.}$$
$$\therefore \quad 104 \times 63 \times 6 \times 10^{-4} \text{ gm. of } Cu.$$
$$i.e., \quad 3{\cdot}9312 \quad ,,$$

In the case of the cupric chloride, we name

$$\tfrac{63}{2} \text{ gm. of } Cu = \text{gm. of } H,$$

and the rest of the data are the same; hence 1·97 gm. of Cu.

EXERCISE LVII.

1. If 78 milligrammes of potassium be thrown into water, 2 milligrammes of hydrogen will be absorbed. What weight of caustic potash will be formed?

2. The equivalents of potassium, rubidium, and caesium being 39, 85, and 133 respectively, what volume of hydrogen at 27° and 1000 mm. will be produced by the action of 100 gm. of each metal upon water?

3. How many cubic centimetres of hydrogen, at the normal temperature and pressure, are evolved, when 39 mg. of potassium and 40 mg. of calcium are thrown into water?

4. The action of sodium on water is represented by the equation

$$Na_2 + 2\,H_2O = 2\,NaHO + H_2;$$

how much water is required to take part in this reaction to obtain 22·38 litres of hydrogen at 0° and 760 mm.

5. How many litres of chlorine gas, at 0° and 760 mm., are required for the exact liberation of the iodine contained in 10 gm. of iodide of potassium?

6. How many litres of hydrogen, at 750 mm. and 25°, can be obtained by dissolving 100 gm. of magnesium in hydrochloric acid?

7. What volume of hydrogen, at 12° and 750 mm., is disengaged when 100 gm. of zinc dissolve in dilute sulphuric acid?

8. How much nitrate of silver can be obtained from one pound of metallic silver, and how much nitric acid (HNO_3) is necessary for the reaction?

9. One kg. of a dilute sulphuric acid is found to dissolve 130 gm. of zinc. Find the percentage of strong acid in the dilute acid.

10. By dissolving 0·4442 gm. of metallic cobalt in an acid, 177·4 c.c. of hydrogen at 10° and 750 mm. are obtained. The specific heat of cobalt is 0·107; calculate its atomic weight.

11. How many grains of carbonic acid will combine with 100 grains of calcic oxide to form calcic carbonate?

12. How many litres of ammonia, at 15° and 740 mm., are required to neu-

tralize 100 gm. of pure sulphuric acid (H_2SO_4), and what is the weight of the neutral salt produced?

13. The analysis of the silver salt of a monobasic acid gave the following numbers—

Carbon, 28·3; Silver, 51·6; Hydrogen, 4·3; Oxygen, 15·8.

Calculate the formula of the salt.

14. How much oxygen can be obtained from 100 gms. of potassium chlorate by heating to a red heat?

15. How many litres of ammonia, at 10° and 75 cm., can be obtained from 100 gms. of pure sal ammoniac?

16. How much marble is required for the preparation of 67·14 litres of carbonic acid at 0° and 760 mm.?

17. What weight, theoretically, of sulphur is required to produce 100 tons of sulphuric acid?

18. How much potassic carbonate is required for the complete precipitation of 100 grammes of calcic chloride?

19. What volume of sulphuretted hydrogen, at 0° and 76 cm., can be obtained from 10 grammes of ferrous sulphide?

20. If 1,000 grains of a potable water contains 0·228 grain of lime, estimated as carbonate, what must be its degree of hardness?

21. If the current of a battery of 10 Grove's cells connected in series is sent through each of two voltameters placed in the circuit containing solutions of cupric sulphate and silver nitrate respectively; how much copper and how much silver will be deposited, while 3·25 gms. of zinc is dissolved in the battery?

22. How much copper and silver would be deposited during the solution of the above quantity of zinc, if the battery were arranged in two parallel series, each of 5 cells, instead of in a single series of 10 cells?

23. The same electric current is passed successively through solutions of cuprous chloride and cupric chloride. How much metal is in each solution precipitated for every 500 c.c. of chlorine evolved?

APPENDIX.

ART. 42.—The proposal of the Chancellor of the Exchequer, mentioned at page 49, has been carried.

ART. 46.—Switzerland and Italy have agreed to leave the Latin Union, and to adopt a gold standard.

PAGE 104.—An International Conference, which met at Washington, has decided to adopt the meridian of Greenwich for the origin of longitude.

ANSWERS TO THE EXERCISES.

EXERCISE I.

1. $20 \cdot 8\dot{3}$£ ; 938,880 farthings.
2. 197·708 shillings; £61 14*s.* $6\frac{3}{4}d.$
3. 2·779£ ; 8·073£.
4. £12 3*s.* $8\frac{3}{4}d.$; £17 18*s.* $3\frac{1}{2}d.$
5. £11 3fl. 7c. 7m.; £7 5fl. 3c. 7m.; £8 7fl. 7c. 5m.
6. £7 10*s.* 9*d.*; £45 7*s.* $11\frac{3}{4}d.$
7. $3\frac{1}{4}d.$; 2*s.* $9\frac{3}{4}d.$; £1 8*s.* $4\frac{1}{2}d.$; £14 3*s.* $10\frac{1}{4}d.$
8. 2·5 fl.; 1·25 fl.; 25 mils; 16·6 mils; 12·5 mils.
9. 1*d.*; $2\frac{1}{2}d.$; $4\frac{3}{4}d.$; 2*s.* 6*d.*; 4*s.*
10. £45 7*s.* $7\frac{3}{4}d.$
11. 98·878£ ; 1756·44 shillings.

EXERCISE II.

1. 3,600£.
2. £14 3*s.* $10\frac{1}{4}d.$
3. £337 10*s.*
4. £1 19*s.* 5*d.*
5. 52,894,143 letters ; £330,588 7*s.* $10\frac{1}{2}d.$
6. £2,067 7*s.* $8\frac{1}{2}d.$
7. £400 19*s.* $0\frac{3}{4}d.$
8. 10£ per ox, and 15£ per cow.
9. 30 men.
10. 7 months.
11. 273 miles.
12. 21 shillings.
13. 10 pence per lb.
14. 50 shillings per 100 loads.
15. 9 pence per dozen.
16. 2 shillings per dozen.
17. 2*s.* $2\frac{2}{3}d.$ per cwt.
18. $6\frac{1}{2}$ to 7 pence per lb.
19. 5*s.* per first-class passenger, and 208 passengers; 3*s.* 9*d.* per second-class passenger, and 304 passengers; 2*s.* 6*d.* per third-class passenger, and 488 passengers.
20. *A* had 9 units of money for every 2 that *B* had.
21. 3 lbs. of butter.
22. No.
23. £74 11*s.* per ton.

EXERCISE III.

1. £3 2*s.* 8*d.*
2. 3·7 per cent.
3. 666 quarters.
4. £156 ; 120 per cent.
5. £12 13*s.* + per head.
6. 13 shillings.
7. 800 lbs.
8. 27*s.* $7\frac{1}{2}d.$
9. 1 per cent. loss.
10. 48·6 per cent.; 182 yards.
11. 400 acres.
12. 30 per cent loss.
13. 18£.
14. 350 apples.

EXERCISE IV.

1. 100 gallons.
2. 37 + per cent.
3. 84·3 lbs. nitre; 15·8 lbs. charcoal; 11·8 lbs. sulphur.
4. £1 6*s*. 8*d*. per acre.
5. 3 gallons of water per 20 gallons of pure milk.
6. 5 lbs. of 2nd per 7 lbs. of 1st; 19 lbs of 3rd per 7 lbs. of 1st.
7. £263 10*s*. 10*d*.
8. 5 shillings in the pound.
9. 15 shillings per pound.
10. 11*s*. 6*d*. per pound.
11. $110\dfrac{m-\dfrac{pa+qb+rc}{a+b+c}}{m}$ per cent.
12. $\left(1+\dfrac{k}{100}\right)\dfrac{ap+bq+cr+ds}{a+b+c+d}$ shilling per lb.
13. $(l+m+n)(p+q+r) = 3(lp+mq+nr)$.
14. 44 peaches = 9 shillings.
15. 66 per cent.
16. 2 lbs. of the cheap tea to 1 lb. of the dear.
17. £7 10*s*. to A; £9 to B; £7 4*s*. to C.

EXERCISE V.

1. 875£.
2. 1,200£ at 10 %; 1,800£ at 5 %.
3. £616 14*s*.
4. 5 years.
5. £4 16*s*. 4*d*.
6. £2 12*s*. 6*d*.
7. 5 months.
8. 36¼ per cent. per year.
9. 6¼ per cent. per year.
10. £9 8*s*. profit.
11. He borrowed at 5 per cent. and lent at 5½ per cent.
12. 6½ per cent. per year.
13. 4¾ per cent.
14. £983 13*s*. 4*d*.; £983 12*s*. 1½*d*.
15. £2,151 9*s*. 6*d*.
16. £1,847 9*s*. 2*d*.
17. £23 6*s*. 8*d*.; £71 4*s*. 6d.
18. £2 8*s*. 6*d*.
19. £242 2*s*.
20. 4*s*. 9*d*.
21. 906£.
22. 24th September.
23. 33 per cent.

EXERCISE VI.

1. 56*l*. 14*s*.; 437*l*. 11*s*. 8*d*.
2. 41*l*. 1*s*. 0*d*.
3. The former, by 3*s*. 6*d*.
4. 231*l*. 10*s*. 6*d*.
5. 421*l*. 17*s*. 2*d*.
6. 133*l*. 2*s*. 10*d*.
7. 15 years.
8. 2,252 million pounds.
9. 249*l*. 17*s*.
10. 126*l*. 2*s*.; 108*l*. 18*s*. 7*d*.
11. 124*l*. 2*s*. 7*d*.
12. £2,051 and £1,949.
13. £1,000.
14. 26-years.
15. 497*l*. 14*s*.; 381*l*. 9*s*.
16. £1,753.

17. 13,077 dollars.
18. £3,133.
19. 42 years.
20. $6\frac{1}{4}$ per cent.
21. £3,291.
22. £100,000.
23. 32 years.

EXERCISE VII.

1. £1,041 13*s*. 4*d*.; £41 13*s*. 4*d*.
2. £324 per year.
3. £9,776.
4. $97\frac{1}{8}$£ cash = 100£ stock.
5. £80 10*s*. 5*d*. lost; £3 4*s*. per year less.
6. £25 additional per year.
7. $133\frac{1}{3}$£ cash per 100£ stock.
8. £64 12*s*. less.
9. £7 10*s*.
10. $90\frac{5}{13}$.
11. 106£ per share.
12. $32 per year.
13. $2\frac{6}{7}$ years.
14. £5 additional per year.
15. 3·265 per cent.; 3·5 per cent.
16. £3 10*s*. 8*d*.; £1 2*s*. 1*d*.
17. £3 6*s*. 8*d*. in both cases.
18. £210.
19. 6·64 per cent.
20. $6\frac{1}{2}$ per cent loss.
21. 2·7 per cent.
22. £5 gain.
23. £2,967 per week.
24. £2,600,000.
25. £12,240,000.

EXERCISE VIII.

1. £182 7*s*. 2*d*.
2. halfpenny = 1 cent., penny = 2 cents, threepenny = 6 cents, fourpenny = 8 cents, shilling = $24\frac{1}{4}$ cents, florin = $48\frac{1}{2}$ cents, half-crown = $60\frac{1}{2}$ cents, half-sovereign = 2$ $42\frac{1}{2}$ cents, sovereign = 4$ 85 cents.
3. £8,219 4*s*.; 207,288 francs.
4. 5·182 franc per dollar; ·810 mark per franc.
5. 1,977 francs 53 centimes.
6. £39,651,000.
7. £48 18*s*.
8. £216 13*s*. 4*d*.
9. 923·08 roubles.
10. £2,056 11*s*. 2*d*.
11. 90·7 roubles = 100 florins.
12. 360,000 francs.
13. $5\frac{1}{6}$ per cent.
14. $948.
15. 113·001 grains.
16. 1,000 pounds troy.
17. 155.
18. $4{\cdot}8\frac{2}{3}$$ per £.
19. £872.

EXERCISE IX.

1. 87·49 yards; 36·58 metres.
2. 516·6 km.
3. $1{\cdot}5\dot{1}$ link = foot.
4. 36 hundredths of English inch.
5. 16,822 Parisian feet.
6. 2,709 cm.

7. 762 English feet.
8. 150 feet.
9. 6 fr. 20 c. per metre.
10. 11/10, 12/11, 359/329, 2525/2314 metre per yard.
11. 1/10560 inch scale = inch real.

EXERCISE X.

1. 7·2 degrees ; 8 grades.
2. 63·662 grades ; $1/\pi = \cdot31831$.
3. 19·577 yd.
4. 2 radians.
5. ·7854 radians ; 257·8 degrees.
6. 326° 15′; 236° 15′.
7. 13 points ; 146¼ degrees.
8. 7908 + miles.
9. 69·05 statute miles = degree.
10. 1/3 ∟ rise = ∟ span.
11. 66 feet.
12. 6·4 yards ; 14·3 yards.
13. 727 feet.
14. 40 feet.
15. 246·8 miles.
16. ·5236 radian per mile arc ; 1·9 mile radius.
17. 1·852 kilometre = sea-mile.
18. ·9 inch longer.

EXERCISE XI.

1. 151025, 78909, and 82243 sq. km.
2. 39·370 inches = metre.
3. 836 sq. metres.
4. 518 million persons.
5. 124 fr. 52 c. per hectare.
6. 7·48 francs per sq. metre.
7. 83⅓ acres.
8. 760,320 blocks.
9. 1,452 feet.
10. 9,876 feet ; 34,566 feet.
11. 44 feet long by 33 feet broad.
12. 123·45 feet × 246·9 feet.
13. 69·57 yards.
14. 417 feet.
15. ·395 mile.
16. 35·43 poles.
17. 5 feet 3 inches.
18. 61 feet 6 inches.
19. 1*s.* 4*d.* per square foot ; 6*d.* per foot long.
20. 7¼ cents per yard.
21. 3 feet square ; 14 feet square.
22. 16 rounds.
23. £861,980.
24. 2783 yards; 3080 yards.

EXERCISE XII.

1. 22·96 acres.
2. 2·0234 hectares.
3. 121 yards ; 40 yards.
4. 20 acres.
5. 53,044 square yards.
6. 330 square feet.
7. 30 acres.
8. 4·33 acres.
9. £6 1*s.* 2*d.*
10. 427 square feet.
11. 424·12 mm.; 14314 square mm.
12. 191 cm.; 28,652 square cm.

13. 265 dekametres; 832·52 dekametres.
14. 28·274 square inches; 80·214 circular inches.
15. ·35 square inch.
16. 222 square feet.
17. 29,452 square feet.
18. 43·7 acres.
19. 6·062 square miles.
20. $2\frac{1}{2}$ mile per inch.
21. 16,000 sq. ft. = sq. inch.
22. 1/13200 inch per inch; 4 tenths foot per mile.

EXERCISE XIII.

1. 28·32 litre = cubic foot.
2. 103·15 Winchester bushels = 100 Imperial bushels.
3. 18·708 gallons; 245·34 litres.
4. 1 gill nearly.
5. 643·8 bushels; 157 hectolitres.
6. 34*s*. 6*d*. per quarter.
7. ·4125 francs per litre.
8. 1·344 farthings per hour.
9. $\sqrt[3]{a^2/b}$ feet.
10. 16,384 bricks.
11. $34\frac{2}{3}$ cubic inches.
12. 15·9 francs per hectolitre.
13. 22 pence per gallon.
14. 1·598 too little per 100 calculated, 3·222 too little per 100 calculated, 4·871 too little per 100 calculated.
15. 10·079 inches.

EXERCISE XIV.

1. 1·18 cubic feet.
2. 3·1416 square yards.
3. 4·5 feet.
4. 17 – feet.
5. 9 square feet.
6. 268 thousand million cubic inches; 1,331 thousand
7. $1\cdot76 \times 10^4$ cubic feet.
8. 2 ∟ altitude = ∟ radius.
9. 4·57 feet.
10. $\sqrt{6}/\sqrt{\pi}$ V of sphere = V of cube.
11. 1/1600 square inch = square ft.; 1/64000 cubic inch = cubic ft.
12. $144\pi^2$ cubic inches; $36\pi\{1+\sqrt{4\pi^2+1}\}$ sq. in.
13. 62,204 cubic feet.
14. 188·5 cubic inches.

EXERCISE XV.

1. 365·24 + mean solar days.
2. $4^h\ 30^m$.
3. 2 16 A.M., 0 24 A.M., 7 20 P.M., 1 22 P.M., 4 35 A.M.
4. 5,000 men.
5. $2\frac{1}{7}$ hours.
6. 4 to 3.
7. 22·3 minutes.
8. $5\frac{1}{3}$ days.
9. 14 minutes.
10. 10·9 hours.
11. A takes $2\dfrac{pqr}{rp+pq-qr}$ days; and similarly for B and C.
12. 14·8 days.
13. 30 minutes 1 second later.
14. 1/4, 7/29, 8/33, 31/128, 132/545 days.

EXERCISE XVI.

1. $1{\cdot}4\dot{6}$ ft. per sec.; $\cdot 68\dot{1}\dot{8}$ mile per hour.
2. 88 feet per second.
3. $2{\cdot}99 \times 10^8$ metres per second.
4. $27{\cdot}\dot{7}$ cm. per second.
5. 186,000 miles per second.
6. 91 miles by former = 81 miles by latter.
7. 12·18 nautical miles per hour.
8. 22·7 miles per hour.
9. 1·40 metres per second.
10. 6 17 metres per second.
11. 11 to 13, 22, 55 to 66, and 88 miles per hour.
12. ·14687, ·244, ·45458, ·463, and 2·048 kilometres per second; 14687, 24400, 45458, 46300, and 204800 centimetres per second.
13. 5·44 miles per hour.
14. 18·1 miles per hour
15. 1 hour 20 minutes.
16. $8\frac{1}{2}$ hours.
17. 4 miles per h., $3\frac{1}{2}$ miles per h.
18. 4 minutes.
19. $abc/(b+c)$ miles.
20. 195 miles.
21. 17·8 knots per hour; 20·5 miles per hour.

EXERCISE XVII.

1. 60 miles.
2. 1,854 miles.
3. *A* takes $16\frac{1}{2}$, *B* takes $13\frac{3}{4}$ hours.
4. 4·29 miles per hour.
5. In 8 hours, at 38 miles distance.
6. 1 minute.
7. By 106 yards.
8. 970 yards.
9. 5·3 yards.
10. 19 yards.
11. 4 minutes $50\frac{1}{2}$ seconds.
12. *A*, by $21\frac{1}{2}$ yards.
13. *A* 160, and B 120 yds. per min.
14. 20 miles per hour.
15. $7^h\ 12^m$, A.M.
16. 90 miles.
17. 7 hours nearly; 20 hours nearly.
18. 20 cars per hour meet; 4 cars per hour overtake.
19. 6 miles per hour; $\frac{1}{2}$ mile.
20. 1 division of vernier = ·052 inch.
21. 1 division of vernier = 19 mins.

EXERCISE XVIII.

1. 2 hours 11 minutes.
2. 21 miles, 2 miles per hour.
3. $4\frac{1}{4}$ hours.
4. $30\frac{a+b}{ab}$, and $30\ \frac{b-a}{ab}$ miles per hour.
5. 28 feet *W.* 30° *S.* per second.
6. 8·94 knot *N.* 26° *E.* per hour.
7. 8·7 min.; up the stream at 30°.
8. 200 feet.
9. 15·7 knots per hour, from between *NW.* and *WNW.*
10. 15·7 knots per hour from *W.* 26° *N.*

EXERCISE XIX.

1. ·10472 radian per sec. = rev. per min.
2. 10 minutes 55 seconds.
3. 75 degrees, $83\frac{1}{3}$ grades, 1·309 radians.
4. 1 min. $1\frac{1}{59}$ sec. after XII.
5. $(n+2)/11$ hours after n o'clock ; IX.
6. 24 seconds.
7. 5·6 minutes past XII.
8. At *W*.
9. 9 hours.
10. 4 yards and 5 yards.
11. 7·5 hours.
12. 5 : 112 : 2800.
13. $15'$ = min.; $15''$ = sec.
14. 7·3 feet per second.
15. 25,000 miles; 520 miles per hour.
16. 740 miles per hour.
17. 12 miles per hour.
18. 1·3 minutes gained per hour.
19. 8 hours 45 minutes.
20. 9 miles ; $4\frac{1}{2}$ hours.

EXERCISE XX.

1. 18 km. per min. per min.
2. 200 yards per min., 12,000 yards per min. per min.
3. L and T; *mg*.
4. 38,640 yards per min. per min.
5. 197 feet.
6. 629 feet.
7. 5 seconds.
8. 126·5 ft. per sec.
9. 6·21 feet; 155 feet.
10. 2 5 seconds.
11. 1·25 second.
12. 370 feet; 2,318 feet of fall.
13. 5·37 feet per sec.
14. 864 feet; 464 feet per sec.
15. 161 feet per sec.; 402 feet.
16. 225 feet ; $2\frac{1}{2}$ sec.
17. 98 feet per sec.
18. 121 feet.
19. $2\frac{1}{4}$ sec.; 42 feet.
20. 10,800 yards per min. per min.
21. 2·5 feet backwards per sec.
22. 3 feet per sec. per sec.

EXERCISE XXI.

1. 5·5 sec.; 87·6 feet per sec.
2. 8·05 feet ; $120\frac{3}{4}$ feet.
3. 2 feet per sec. per sec.
4. 4·4 feet forwards; ·16 feet downwards.
5. 100 feet.
6. 210 feet.
7. $3\frac{1}{4}$ sec.; 325 feet.
8. 113 feet per sec.
9. 4 to 1.
10. 720 feet per sec.
11. 2 sec.; 147 feet.
12. $10/\pi$ feet per sec.
13. $1600\pi^2$ cm. per sec. per sec.; 80π cm. per sec.

EXERCISE XXII.

1. 1·21528 pound troy; ·91146 oz. troy.
2. 48·5 lb.; 11·34 kgm.
3. ·984 cwt.; ·984 tons.
4. $37\frac{1}{3}$ shillings per cwt.; $37\frac{1}{3}$£ per ton.
5. 4·34 pence per lb.
6. 39·688 shillings per cwt.
7. 15 fr. 28 ct. per kilogramme.

EXERCISE XXIII.

1. 50·8 kgm. = cwt.
2. 351,860,000 cubic feet; 21,991,250,000 lbs.
3. 62,099 cubic inches.
4. 174 lb. per cubic ft.
5. 2,760 kgm.
6. ·016019 gm. per cc.
7. ·01602 cubic ft. per lb.
8. 30,000 cubic feet.
9. 15,274 tons.
10. 23·04 ounces.
11. ·028 ton per cub. ft.; 432 lb. per sq. inch per ft.
12. 4,752 tons.
13. 292 inhabitants per sq. mile.
14. 739 lb.
15. 42.
16. 6·7 lb. per fathom.
17. 40, 10, 2·64.
18. 265,000 bricks.
19. 145,030,000 lb. per sq. mile per inch; 226,600 lb. per acre per inch.
20. 2·2136 lb.
21. 44·27 cwt.

EXERCISE XXIV.

1. 10·27 lb.
2. 9,800 cubic inches of oxygen = cubic inch of mercury.
3. 81 to 16.
4. 25·65 ounces.
5. ·9 cubic foot.
6. ·92, ·94, 1·14, 1·46.
7. 5·66 and 5·34 lb. per 100 lineal feet.
8. ·97.
9. $\dfrac{s-m}{m-1}$ **V** of water per **V** of pure milk.
10. 857 cubic cm.
11. 12*s*. 10*d*. per gallon.
12. 3 **V** of silver per **V** of copper.
13. 7·5 oz. gold; 2·5 oz. quartz; 1 **V** of gold per 3 **V** of quartz.
14. 15 cc.
15. 1 **V** of first per **V** of second.
16. $\dfrac{s_1v_1+s_2v_2+\ldots+s_n v_n}{v_1+v_2\ +\ldots+\ v_n}$.
17. 716 gm. per cc.
18. 1/30 sq. inch nearly.
19. 29·5 lb.; 356 lb.
20. 5·1 ft.
21. 6 and 1.

EXERCISE XXV.

1. At 4 inches from centre of smaller.
2. At the centre; at five-sevenths of the line joining the corner with the centre.
3. Mid point of central piece.
4. In the principal piece at one-fourth of its length from the cross piece.
5. Four tenths of the perpendicular from the mid point of the central piece to the cross piece; five-fourteenths of the central piece from its extremity.
6. Along axis at $(\sqrt{3}+1)/3$ inch from the mid point of the lower piece.
7. 3·24 inch above base in axis.

EXERCISE XXVI.

1. 5 to 8.
2. 750 feet.
3. 64 ft. per sec.; 10,752 lb. by ft. per sec.
4. 1,421 lb. by ft. per sec.
5. 694 ft. upwards per sec.; 6,940 lb. by ft. per sec.; 272 ft. downwards per sec.; 2,720 lb. by ft. per sec.
6. 183 lb. by ft. per sec. per cub. ft.
7. 64 to 5.
8. 63,000 lb. by ft. per sec.
9. 3·1 ft. per sec.
10. 1·2 ft. per sec.

EXERCISE XXVII.

1. 9,800 cm.; 980 cm. per sec.
2. 96 poundals.
3. 12·9 ounces.
4. 161 poundals; 5 lb.
5. 5,000 sec.
6. 51 lb. weight.
7. 16 ft. per sec.; 8 ft.; 960 ft.
8. 90,000 ft.
9. 1/120 poundal.
10. $1/\sqrt{32 \cdot 2}$ sec.
11. 3 M per 2 M.
12. 16·1 ft. per sec.
13. 980 cm. per sec. per sec.
14. 3·8 oz.
15. ·355 oz.; ·5 ft.
16. ·976 ft. per sec. per sec.; 33·2 poundals.
17. 32 poundals.
18. 6 sec.
19. 3·22 ft. per sec. per sec.; 40·25 feet.
20. 1·25 lb. weight; 6 ft. per sec. per sec.

EXERCISE XXVIII.

1. Weight diminished to $\frac{11}{16}$.
2. 44 poundals.
3. 1·093 lb.
4. $\sqrt{3} : 1 : 2$.
5. 2·7 dynes; 2·34 dynes.
6. 17·4 dynes.
7. About 67°.
8. 9·17 lb. weight.

9. 10 pounds.
10. By 2·5 ton weight.
11. 3 lb. weight.
12. 8 lb. and 6 lb.
13. 150 lb. nearly.
14. 60°.
15. 50·6 ft. per sec.
16. 107, 61, and 81 lb. weight.

EXERCISE XXIX.

1. 197 lb. weight.
2. 210 to 1.
3. 94·75 poundals.
4. 900 poundals.
5. $11/15g$. ton weight.
6. 5,475 poundals.
7. 10,300 poundals.
8. 98·97 lbs.
9. ·72 ft.
10. 84°.
11. ·584 dyne per gramme.
12. 60π poundal per foot.
13. At 3/8 and at 4/9 of rod from larger mass.
14. 8·6 inch.
15. ·27.
16. 1·13 radian per sec.
17. $720\pi^2$ and $144\pi^2$ poundals.
18. 10·8 poundals.
19. 264 gm. weight.
20. Weight and tension of $5\sqrt{3}$ lb.
21. 10·4 inch.
22. 993·9 mm.
23. 32·24 poundals per lb.; ·12 beat per sec.
24. 19·94 ft.
25. 39 yds.
26. 31·78 ft. per sec. per sec.
27. 43 sec.
28. 8·2 turns.

EXERCISE XXX.

1. 8·86.
2. 2·98; 4·76 gm.
3. 2,080 grain weight.
4. 17·62 gm.
5. 9·011 cubic inches.
6. ·89 oz. olive oil = oz. sea water.
7. 1·42.
8. 23·68 cubic inches.
9. 2·4 cubic feet.
10. ·4 inch in lower.
11. ·0819 inch.
12. ·875; ·97.
13. 257 ft.
14. 10·4 gm.
15. Neither.
16. 2·5.
17. ·5, 1·25.
18. ·2
19. 10·73 inch in side.
20. 1 to 10.
21. 6·2 lb. in mercury; 5·8 lb. in water.
22. 5 inches.
23. 132 cubic inches.
24. $m\,(1-s)/s$.
25. 32,000 cubic feet.
26. 2·6.
27. 12 V of silver = 5 V of copper.

EXERCISE XXXI.

1. 479 and $6 \cdot 9 \times 10^4$ dynes per sq. cm.
2. 120 lb. weight.
3. 152 lb. weight.
4. 650·23, 678·17, 706·11 mm.
5. 53·53 inch of water.
6. 8×10^5 dynes per sq. cm.
7. 10,330 kgm. per sq. metre.
8. 89 lb. per sq. inch.
9. 174 lb. weight.
10. 13·49 lb. per sq. inch.
11. 69 feet.
12. 2·63 cm.
13. 31·7 feet.
14. ·15 inch.
15. 17·5 lb.
16. 448 feet.
17. ·41 lb. weight per sq. inch.
18. 9,375 lb. weight.
19. 2,625 lb. weight.
20. 93,750 and 132,200 lb. weight.
21. 1·2 kgm.
22. 656·25 lb.
23. 929 kgm.
24. 37·4 lb. weight.

EXERCISE XXXII.

1. 5·15 miles.
2. 4 and 36 inches.
3. 8·2 grains.
4. 22·5 cubic feet.
5. 66 feet.
6. Constant.
7. ·000078 cubic inch.
8. 17·3 cubic inches.
9. 68 cubic inches.
10. 50·7 cm.
11. 1·72 feet.
12. $1/3\sqrt{3}$ atmosphere.
13. ·43 of original pressure.
14. ·62 **M** per **V** = **M** per **V** original.
15. ·68 **M** = **M** original.
16. 5·4 feet.

EXERCISE XXXIII.

1. 7·23308 foot-pounds; ·138254 kilogrammetre.
2. 405,000 foot-pounds.
3. 3,360 foot-pounds.
4. 12·5 foot-pounds; 2·5 lb. weight.
5. $4 \cdot 55 \times 10^5$ foot-pounds.
6. $4 \cdot 44 \times 10^5$ foot-pounds.
7. $1 \cdot 983 \times 10^7$ foot-pounds.
8. 36,945 foot-pounds.
9. 1,200 foot-pounds.
10. 24 ft.-lb.
11. 7,516 ft.-lb. per stroke.
12. 32·2 lb.

EXERCISE XXXIV.

1. 24,000 foot-poundals.
2. 64 to 125.
3. 27,000,000 foot-poundals.
4. 36,064 foot-poundals.
5. 131,250 foot-poundals.
6. 431,500 foot-poundals.
7. 22,065,120 foot-poundals.
8. 180,320 foot-poundals.

9. 834,783 foot-pounds.
10. 1,248 foot-poundals.
11. 2·03 ton weight.
12. $2{\cdot}409 \times 10^7$ foot-poundals; 1·643 poundals.
13. 21,683 poundals.
14. 66·8 feet.
15. 6,666 poundals.
16. 2,332 lb. weight.
17. 1·49 feet.
18. 8·5 lb. weight.
19. 2,067 feet.
20. 169,165 poundals.
21. 181 ft.
22. $ph/(q-p)$ feet.
23. 9·3 ft. per sec.; 654 ∠660.
24. 607 kilogrammetres.
25. 17,578 poundals.
26. $(\sqrt{3}-1)/1$.
27. 1,690 ft. per sec.
28. $6{\cdot}25 \times 10^{12}$ ergs; $3{\cdot}125 \times 10^{11}$ ergs.
29. 1,160 pound by foot square.
30. $2{\cdot}93 \times 10^5$ foot-pounds.
31. 40,894 foot-poundals.
32. 18,867 foot-pounds.
33. 1 ft.; 2 ft. per sec.
34. 33·7 radian per sec.
35. 2·84 feet.
36. $\sqrt{\frac{3g}{l}}$

EXERCISE XXXV.

1. ·987 horse-power.
2. 1·014 force de cheval = horse-power.
3. 4,562 kilogrammetre per min.
4. 6·8 horse-power.
5. 110,880,000 ft.-lb.
6. $1{\cdot}98 \times 10^7$ ft.-lbs; 120 horse-power.
7. 67,200 ft.-lb.; 4·1 horse-power.
8. 89·6 gallons.
9. 146 horse-power.
10. 4 horse-power.
11. 21 inch.
12. 58·4 revolutions per minute.

EXERCISE XXXVI.

1. 7° approx.
2. 46·7 lb. weight.
3. 222 lb. weight.
4. 3,240 lb. weight.
5. 19·4 revs. per sec.
6. 7,634 lb. weight.
7. 1·39 lb. weight.
8. $np/101$ knots per hour.
9. 100 lb. weight.
10. 44 to 7.
11. 3 lb. weight.
12. 20 lb. weight.
13. ·5 cwt.; 18 ft.
14. 1 cwt.
15. 1 F, power = 7 F, weight.
16. 1 to 30.
17. 1,209 lb. weight.
18. 22 lb.; 2,400 foot pounds.

EXERCISE XXXVII.

1. 26,880 inch-pound.
2. 3/2.
3. 415·4 grains.
4. 2,916 gm.; 127/125.

5. 112·88 lb.
6. 23·21 inches.
7. 840 lb.
8. 70 lb.
9. ·9 inch in front of axle.
10. 24,500 lb.
11. 11·8 lb. weight per sq. inch.
12. 10·5 lb. weight per sq. inch.
13. 4 ft.
14. 7·6 ft. from end.
15. 8 and 9 lb. weight.
16. 1·6 ft. from fulcrum towards *A*.
17. At the fulcrum.
18. 2 ft. from other end.
19. At 3 ft. from end, and at its mid point.
20. 5 ft. from *A*.
21. 4 lb. and 6 lb.
22. 14 lb.
23. $2\sqrt{2}$ lb. weight.
24. 20 lb.

EXERCISE XXXVIII.

1. Increased in ratio of 16 to 9.
2. 30 lb. weight.
3. 200 per cent.
4. 2 sec.
5. ·0089 ft. per sec. per sec.

EXERCISE XXXIX.

1. 117·8 degs. Cent.; 180 degs. Fahr.
2. 572 and 392° F.
3. 260, 204·4, 149, 93·3, and 37·8° C.
4. 16·67° C.; 39·1° F.
5. 164, 30, 21·1, and −34·4° C.
6. 332, 412, 467, 1,021, 1,091, and 1,530° C.
7. 169, 95, 675, and 640° F.
8. −38·99, 194, 455, and 2,732° F.
9. −40.
10. −25·6.
11. 41 and 5, 50 and 10, 59 and 15, 68 and 20.
12. 320° F.

EXERCISE XL.

1. $1·06 \times 10^{10}$ ergs.
2. 3·968 lb. by deg. Fahr. = kgm. by deg. Cent.
3. 70° C.
4. 212·5 lb. of water by deg. Cent.
5. 9·07 lb. of water by deg. Fahr.
6. 2·514 lb. of water by deg. Fahr.
7. ·22 deg. Fahr.
8. 67,320,000 ft. lb.; 87,200 lb. of water by deg. Fahr.
9. 1,356,800 metre.
10. 6,116,000 ft.-lb. = lb. of coal.
11. $66\frac{2}{3}$ per cent.

EXERCISE XLI.

1. 96·8° C.
2. 5 degs. Cent.
3. 712° C.
4. 20·5° C.
5. 24·3° C.
6. 283° C.

7. ·114 M of water = M of iron.
8. 5·6 lb.
9. ·096 M of water = M of metal.
10. 20·8° C.
11. ·443 M of water = M of substance.
12. 25·7° C.; 9·12 gm. of water.
13. 157 foot-pounds = lb. of iron by deg. Cent.
14. 5 degs. Fahr.
15. Would be 934 degs. Cent.
16. 100 ft. per sec.

EXERCISE XLII.

1. 79 lb. of water by deg. Cent. = lb. of ice.
2. 2/3 lb.
3. 8 oz.
4. 10 oz.
5. 2·5 lb.
6. 79·28 kgm. of water by deg. Cent. = kgm. of ice.
7. 100° C.
8. 4·4 lb. water and ·1 lb. ice, at 0° C.
9. ·091.
10. 71·5° C.
11. ·47.
12. 1·9 gm.
13. 49·1° C.
14. 7,900 gm.

EXERCISE XLIII.

1. ·061 cubic ft. per cubic ft.
2. 50·024 ft.
3. ·24 inch in excess or defect.
4. 10·0036 yds.
5. ·024 ft.
6. 2 L of brass = 3 L of steel.
7. $4·85 \times 10^4$ lb. of water by deg. Cent.
8. 80 and 81·33 cc.; ·00017 V per V per deg. Cent.
9. ·000074 inch per inch per deg. Cent.
10. ·000347 V per V per deg. Cent.
11. 1/1110.
12. 29·48 inch.
13. 98·64° C.

EXERCISE XLIV.

1. 764 cubic inches.
2. 423 cubic inches.
3. 209·8 cc.
4. 450·5 cubic inches.
5. 130° C.
6. 18·4 litres.
7. 1,020 cubic inches.
8. 459° F.; 219° R.
9. 180·04 gm.

EXERCISE XLV.

1. 4·68 gramme-degree.
2. $4·31 \times 10^6$ gramme-degree.
3. $6·23 \times 10^7$ gramme-degree.
4. $1·89 \times 10^6$ lb. of water by deg. Cent. per hour.
5. $3·06 \times 10^6$ gramme-degree.
6. 28·8 pound-degree per sq. ft. per hour.
7. 1·09 gramme-degree per sec. per sq. cm. = deg. Cent. per cm.
8. 14, 13·6, 64, 13·5.

EXERCISE XLVI.

1. 9·31.
2. 5·736 *C.G.S.* units.
3. 48 oscillations per minute.
4. 49 : 36 : 25.
5. ·0461 ft.-grain-sec. unit = *C.G.S.* unit.
6. 3·47.
7. 1·16 dynes per *C.G.S.* pole.
8. ·00019.
9. 5° 10′.
10. 3·26 gm. by cm.$^{?}$.
11. ·1782 *C.G.S.* unit.

EXERCISE XLVII.

1. 1,000 *C.G.S.* units.
2. 8 to 5.
3. 18·3 and 36·7 units of quantity.
4. 350 *C.G.S.* units.
5. 1 to 5 ; 5 to 1.
6. ·0764 *C.G.S.* units per sq. cm.
7. 100 to 1.
8. 2 W final = 5 W original.
9. 3, 4, 5.
10. 25 to 24.
11. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
12. ·00601 calories.

EXERCISE XLVIII.

1. $\sqrt{10}/100000$ for quantity and current ; 1/100 for capacity.
2. $\sqrt{10}/1000$ for quantity and current ; 100 for capacity.
3. $2·790 \times 10^{-4}$, $8·504 \times 10^{-3}$, $3·281 \times 10^{-2}$.
4. 1/1000.
5. 10 units of current = ampere.
6. 735·75 watts = cheval vapeur.
7. 10 joules = kilogrammetre.
8. ·18 horse-power.
9. 7·31 amperes.
10. ·0195 ampere.
11. ·0379 watt.
12. 20·3 volts.
13. 5 ohms.
14. $9·84 \times 10^{8}$.
15. 7·2 amperes.
16. 3,600,000 joules.

EXERCISE XLIX.

1. 3 to 10.
2. 91 microhms.
3. 384 to 343.
4. 100 to 1.
5. 10·8 ohms.
6. 6·5 ohms per mile.
7. 3·164 metres.
8. 7,682 ohms.
9. $4·5 \times 10^{-8}$ metre.
10. 1,622 *C.G.S.* unit = cm. per sq. cm.
11. $2·72 \times 10^{10}$ miles.
12. 2·1 ohms.
13. 1/109, 1/1009, 1/1099, 1/1108.
14. 4,795 ohms.
15. 16·7 ohms.
16. 5 to 7.
17. 1 to 10.

18. In series.
19. ·048 ampere.
20. In multiple arc.
21. Internal resistance equal to external.
22. ·0015, ·0082, and ·0062 ampere.
23. 1·65 watts.
24. 10 ft. from copper end.
25. 100.

EXERCISE L.

1. 1·1 and 4·9 feet per wave.
2. 1·33 feet per wave.
3. $1\frac{1}{3}$ mile.
4. 134 ft.
5. 4·8, 9·5, and 14·4 sec.
6. 36·3 vibrations.
7. 467 yd.
8. 1,091 and 909 vibrations per sec.

EXERCISE LI.

1. 349 metres per sec.
2. 4·65 secs. after one.
3. 35 inches per vibration; $17\frac{1}{2}$ inches.
4. 512 vibrations per second.
5. 34·4 vibrations per sec.
6. 69 vibrations per sec.
7. 182 vibrations per sec.; by 1 vibration per sec.
8. 118 vibrations per sec.
9. 937 ft. per sec.

EXERCISE LII.

1. 16 to 1.
2. 16 to 9.
3. 1·02 ft.
4. 5 candles.
5. 30 earth-quadrants per sec.
6. $5{\cdot}07 \times 10^{14}$ vibrations per sec.
7. 314×10^6 metres per sec.
8. 9/8.
9. 30°.
10. 28°; will be reflected internally.
11. ·045, ·041, ·077.
12. ·63; 1·30.

EXERCISE LIII.

1. 1 ft.
2. 9 and $4\frac{1}{2}$ inches from mirror.
3. 37° from normal.
4. 6 inches and 3 ft. 6 inches below surface of water.
5. 16 ft.
6. 4 inches.
7. 15 ins. from lens on same side.
8. ·05 and 18 inches.
9. $1\frac{1}{10}$ and 2 ft. on other side, image true; 5 and $\frac{1}{11}$ ft. on same side, image virtual.
10. 1 ft. 6 in.
11. At infinity.

EXERCISE LIV.

1. 39 M of K + 14 M of Fe + 18 M of C + 21 M of N = 92 M of ferrocyanide; 23 M of Na + 32 M of S + 24 M of O = 79 M of hyposulphite.
2. 75 gm.
3. (1) 46·6 M of Fe. 53·3 ,, S. (2) 9·8 M of Mg. 13·0 ,, S. 71·5 ,, O. 5·7 ,, H.
4. (1) 85 M of Hg. 15 ,, Cl. (2) 73·9 M of Hg. 26·1 ,, Cl. (3) 26·7 M of C. 2·2 ,, H. 71·1 ,, O.
5. 76·3 M of Hg + 5·4 M of N + 18·3 M of O.
6. 88·86 M of O + 11·14 M of H = 100 M of H_2O.
7. C_2H_4O.
8. $NaPO_3$.
9. $AgNO_2$.
10. $CuFeS_2$.
11. 400 c.c. of oxygen.
12. 88·9 M of O + 11·1 M of H.
13. CH_2O is empirical formula.
14. 34·7 V of O + 65·3 V of N.

EXERCISE LV.

1. 1·965 gm. per litre.
2. 670 and 694 cc. per gm.
3. 537 gm.
4. 200 cubic inches.
5. 21 V of O + 79 V of N; 23 M of O + 77 M of N; 14·42.
6. 14 M_H per atom.
7. 16 M_H per molecule.
8. 17 M_H per molecule.
9. 28 M_H per molecule.
10. 87·81 M_H per molecule; 43·9 and 45·5.
11. 3·58 gm.
12. $Ag_2C_5H_6O_4$.
13. ·056 M of water = M of silver.
14. ·177.

EXERCISE LVI.

1. 188 gm. of SO_2, 53 gm. of H_2O.
2. 178 gm.
3. 83·8 and 140 litres.
4. 140 litres.
5. 7·23 litres.
6. 36 gm.; 22·3 litres.
7. 60 and 40 volumes.
8. 2 volumes; 36 gm.
9. 600 volumes; CO_2 and H_2O.
10. 11·6 lb.; 153 cubic feet.
11. 14·3 litres; 3·6 gm. of CO_2 and 1·5 gm. of H_2O.
12. 46·43 cubic feet.
13. 26 litres; 10 gm. of O + 21 gm. of SO_2 + 69 gm. of N.
14. C_2H_6; 30.
15. 4 litres; 1·075 gm.
16. 4·7 and 6·9 gm.
17. 8,400 lb. of water by deg. Fahr. = lb. of coal.
18. 7,704 gramme-degrees.
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